RECONSTRUCTION ALGEBRAS OF TYPE A

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Abstract. We introduce a new class of algebras, called reconstruction algebras, and present some of their basic properties. These non-commutative rings dictate in every way the process of resolving the Cohen-Macaulay singularities $\mathbb{C}^2/G$ where $G = \frac{1}{r}(1,a) \leq GL(2,\mathbb{C})$.

1. Introduction

It is not a new idea that non-commutative algebra in many ways dictates the process of desingularisation in algebraic geometry; this has been a theme in many recent papers (eg [Van04a], [BKR01], [Bri06]) however almost all research in this direction has taken place inside the relatively small sphere of Gorenstein singularities. For example when considering rings of invariants by finite subgroups of $GL(n,\mathbb{C})$, the Gorenstein hypothesis forces the subgroups inside $SL(n,\mathbb{C})$.

For G a finite subgroup of $SL(2,\mathbb{C})$ it is well known that the preprojective algebra of the corresponding extended Dynkin diagram encodes the process of desingularising the Gorenstein Kleinian singularity $\mathbb{C}[x,y]^G$. From the viewpoint of this paper, the preprojective algebra should be treated as an algebra that can be naturally associated to the dual graph of the minimal commutative resolution, from which we can gain all information about the process of desingularisation. Thus the preprojective algebra is defined with prior knowledge of the dual graph of the minimal resolution, but since it is Morita equivalent to the skew group ring we could alternatively use this purely algebraic ring. The question arises whether there are similar non-commutative algebras for finite subgroups of $GL(2,\mathbb{C})$.

The answer is yes, and in this paper we prove it for the case of finite cyclic subgroups $G = \frac{1}{r}(1,a) \leq GL(2,\mathbb{C})$ (for notation see Section 2).

For such a group G, we associate to the dual graph of the minimal commutative resolution (complete with self-intersection numbers) a non-commutative ring $A_{r,a}$ which we call the *reconstruction algebra* and prove that $A_{r,a}$ is isomorphic to the endomorphism ring of the special Cohen-Macaulay modules in the sense of Wunram [Wun88]. This is important since it shows that for cyclic groups there is a structural correspondence (via the underlying quiver) between the special CM modules and the dual graph complete with intersection numbers, thus generalizing McKay's observation for finite subgroups of $SL(2,\mathbb{C})$ to finite cyclic subgroups of $GL(2,\mathbb{C})$.

This is a correspondence on the level of the underlying quiver, however if we also add in the information of the relations we get more: in this paper we prove that the reconstruction algebra $A_{r,a}$

- contains all information regarding the singularity (as its centre)
- is finitely generated over its centre, so is 'tractably' non-commutative
- contains enough information to construct the minimal resolution (via moduli spaces of finite dimensional representations)
- contains exactly the same homological information as the minimal resolution (through a derived equivalence)

Although this paper studies cyclic subgroups of $GL(2,\mathbb{C})$ and therefore both the singularities $\mathbb{C}^2/G$ and their minimal resolutions are toric, the ideas in this paper are independent of toric geometry and as such provide the correct framework for generalisation.

We also remark that in general the reconstruction algebra is *not* homologically homogenous in the sense of Brown-Hajarnavis [BH84]. This should not be surprising, as there are many other examples of non-commutative resolutions of sensible non-Gorenstein Cohen-Macaulay singularities which are not homologically homogeneous ([QS06] and [SdB06, 5.1(2)]). Non-commutative crepant resolutions have yet to be defined for Cohen-Macaulay singularities, however when $G \nleq SL(2,\mathbb{C})$ the minimal resolution of $\mathbb{C}^2/G$ is not crepant yet is still important. Hence the rings we produce should certainly be examples of (non-crepant) non-commutative resolutions, whenever such a definition is conceived. The failure of the homologically homogeneous property suggests we ought to again think hard about the non-commutative analogue of smoothness.

In fact the reconstruction algebra $A_{r,a}$ should be the *minimal* non-commutative resolution in some rough sense; certainly there is the following picture of derived categories:

$$\begin{tikzcd} D^b(\mathfrak{coh}X) & & D^b(\mathfrak{mod}\mathbb{C}[x,y]\#G) \\ & D^b(\mathfrak{mod}A_{r,a}) \arrow[ul, "\cong"'] \arrow[ur, hook] & \end{tikzcd}$$

so we should still perhaps view the skew group ring as a non-commutative resolution, just not the smallest one.

In this paper we work mostly in the unbounded derived categories where arbitrary coproducts exist. This allows us to use techniques such as Bousfield localisation and compactly generated categories to simplify some of the work needed to obtain bounded derived equivalences, which in turn saves us from having to prove at the beginning that the reconstruction algebra has finite global dimension.

This paper is organized as follows - in Section 2 we define the reconstruction algebra associated to a labelled Dynkin diagram of type A and describe some of its basic structure. In Section 3 we prove that it is isomorphic to the endomorphism ring of some Cohen-Macaulay modules. In Section 4 the minimal resolution of the singularity $\mathbb{C}^2/\frac{1}{r}(1,a)$ is obtained via the moduli space of representations of the associated reconstruction algebra $A_{r,a}$, and in Section 5 we produce a tilting bundle which gives us our derived equivalence. In Section 6 we prove that $A_{r,a}$ is a prime ring and use this to show that the Azumaya locus of $A_{r,a}$ coincides with the smooth locus of its centre $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$. This then gives a precise value for the global dimension of $A_{r,a}$, which shows that the reconstruction algebra need not be homologically homogeneous.

Throughout we shall use $\mathrm{D}(\mathscr{A})$ to denote the unbounded derived category and $\mathrm{D}^b(\mathscr{A})$ to denote the bounded derived category. When working with quivers, we shall write xy to mean x **followed by** y**.** We work over the ground field $\mathbb{C}$ but any algebraically closed field of characteristic zero will suffice.

The moduli results in this paper have been independently discovered by Alastair Craw [Cra07] using multi-linear technology from toric geometry. The benefits of his approach is that the minimal resolution is produced by using global arguments (as opposed to my local arguments), however the technique here generalizes to the non-toric case. Also, here the non-commutative ring can be explicitly written down. Both approaches have their merits.

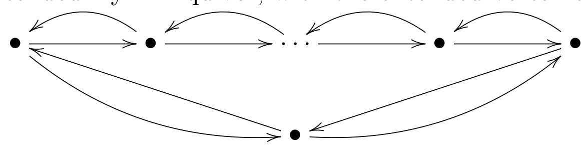
This paper formed part of the author's PhD thesis at the University of Bristol, funded by the EPSRC. Thanks to my supervisor Aidan Schofield for useful discussions. Thanks also to Ken Brown, Iain Gordon, Alastair Craw and Alastair King.

2. The Reconstruction Algebra of Type A

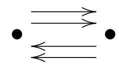
Consider, for positive integers $\alpha_i \geq 2$, the labelled Dynkin diagram of type A:

$$\overset{-\alpha_n}{\bullet} \text{------} \overset{-\alpha_{n-1}}{\bullet} \text{------} \cdots \text{------} \overset{-\alpha_2}{\bullet} \text{------} \overset{-\alpha_1}{\bullet}$$

We call the vertex corresponding to α_i the i^{th} vertex. To this picture we associate the double quiver of the extended Dynkin quiver, with the extended vertex called the 0^{th} vertex:



Name this quiver Q'. For the sake of completeness note that that for $n = 1$ by Q' we mean



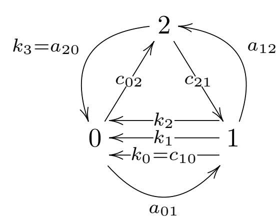
Now if any $\alpha_i > 2$, add an extra $\alpha_i - 2$ arrows from the i^{th} vertex to the 0^{th} vertex. Name this new quiver Q. Notice that when every $\alpha_i = 2$, $Q = Q'$ is exactly the underlying quiver of the preprojective algebra of type $\widetilde{A}_n$.

For convenience label the arrows in Q as follows:

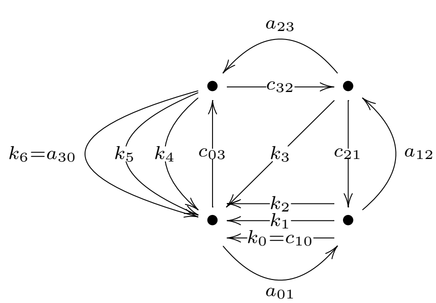
if $n = 1$	label the 2 arrows from 0 to 1 in Q' by a_1, a_2
	label the 2 arrows from 1 to 0 in Q' by c_1, c_2
	label the extra arrows due to α_1 by $k_1, \dots k_{\alpha_1 - 2}$
if $n \geq 2$	label the clockwise arrows in Q' from i to $i-1$ by c_{ii-1} (and c_{0n})
	label the anticlockwise arrows in Q' from i to $i+1$ by a_{ii+1} (and a_{n0})
	label the extra arrows by $k_1, \dots, k_{\sum(\alpha_i - 2)}$ anticlockwise

It is also convenient to write A_{ij} for the composition of anticlockwise paths a from vertex i to vertex j, and similarly C_{ij} as the composition of clockwise paths, where by C_{ii} (resp. A_{ii}) we mean not the empty path at vertex i but the path from i to i round each of the clockwise (resp. anticlockwise) arrows precisely once. For reasons which become obvious in the following examples, it is sensible to denote $c_{10} := k_0$ and $a_{n0} := k_{\sum(\alpha_i - 2)+1}$.

Example 2.1. For $[\alpha_1, \alpha_2] = [4, 2]$ the quiver Q is



Example 2.2. For $[\alpha_1, \alpha_2, \alpha_3] = [4, 3, 4]$ the quiver Q is

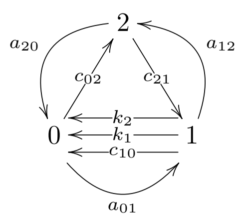


Denote by l_r the number of the vertex associated to the tail of the arrow k_r and denote $u_i := \max\{j : l_j = i\}$ and $v_i := \min\{j : l_j = i\}$. Because we have defined $k_0 := c_{10}$ and $k_{\sum(\alpha_i-2)+1} := a_{n0}$ it is always true that $v_1 = 0$ and $u_n = \sum(\alpha_i - 2) + 1$. For $2 \leq i \leq n$ write $V_i := \max\{j : l_j < i\}$ and set $V_1 := 0$. In the above example $u_1 = 2$, $v_3 = 4$, $V_{l_5} = V_3 = 3$ and $V_{l_2} = V_1 = 0$.

Definition 2.3. *For labels $[\alpha_1, \ldots, \alpha_n]$ with each $\alpha_i \geq 2$, define the reconstruction algebra of type A as the path algebra of the quiver Q subject to the following relations:*

$$\begin{array}{llll}
\textit{if } n=1 & c_2a_1 = c_1a_2 \textit{ and } a_1c_2 = a_2c_1 & & \\
 & k_1a_1 = c_2a_2 \textit{ and } a_1k_1 = a_2c_2 & & \\
 & k_ta_1 = k_{t-1}a_2 \textit{ and } a_1k_t = a_2k_{t-1} \ \forall\ 2 \leq t \leq \alpha_1 - 2. & & \\
\textit{if } n \geq 2 & \textit{Step 1:} & \textit{If } \alpha_1 = 2 & c_{10}a_{01} = a_{12}c_{21} \\
 & & \textit{If } \alpha_1 > 2 & k_sa_{01} = k_{s+1}C_{01}, a_{01}k_s = C_{01}k_{s+1} \ \forall\ 0 \leq s < u_1 \\
 & & & k_{u_1}a_{01} = a_{12}c_{21}. \\
 & & & \vdots \\
 & \textit{Step t:} & \textit{If } \alpha_t = 2 & c_{tt-1}a_{t-1t} = a_{tt+1}c_{t+1t} \\
 & & \textit{If } \alpha_t > 2 & c_{tt-1}a_{t-1t} = k_{v_t}C_{0t}, C_{0t}k_{v_t} = A_{0l_{V_t}}k_{V_t} \\
 & & & k_sA_{0t} = k_{s+1}C_{0t}, A_{0t}k_s = C_{0t}k_{s+1} \ \forall\ v_t \leq s < u_t \\
 & & & k_{u_t}A_{0t} = a_{tt+1}c_{t+1t} \\
 & & & \vdots \\
 & \textit{Step n:} & \textit{If } \alpha_n = 2 & c_{nn-1}a_{n-1n} = a_{n0}c_{0n}, c_{0n}a_{n0} = A_{0l_{V_n}}k_{V_n} \\
 & & \textit{If } \alpha_n > 2 & c_{nn-1}a_{n-1n} = k_{v_n}c_{0n}, c_{0n}k_{v_n} = A_{0l_{V_n}}k_{V_n} \\
 & & & k_sA_{0n} = k_{s+1}c_{0n}, A_{0n}k_s = c_{0n}k_{s+1} \ \forall\ v_n \leq s < u_n
\end{array}$$

Example 2.4. The reconstruction algebra of type A associated to $[4, 2]$ is



$$\begin{array}{lll}
k_2a_{01} = a_{12}c_{21} & c_{10}a_{01} = k_1c_{02}c_{21} & k_1a_{01} = k_2c_{02}c_{21} \\
c_{21}a_{12} = a_{20}c_{02} & a_{01}c_{10} = c_{02}c_{21}k_1 & a_{01}k_1 = c_{02}c_{21}k_2 \\
c_{02}a_{20} = a_{01}k_2 & &
\end{array}$$

Example 2.5. The reconstruction algebra of type A associated to $[4, 3, 4]$ is the path algebra of the quiver in Example 2.2 subject to the relations

$$\begin{array}{rlrlrl}
c_{10}a_{01} &= k_1c_{03}c_{32}c_{21} & a_{01}c_{10} &= c_{03}c_{32}c_{21}k_1 & & \\
k_1a_{01} &= k_2c_{03}c_{32}c_{21} & a_{01}k_1 &= c_{03}c_{32}c_{21}k_2 & & \\
k_2a_{01} &= a_{12}c_{21} & c_{21}a_{12} &= k_3c_{03}c_{32} & c_{03}c_{32}k_3 &= a_{01}k_2 \\
k_3a_{01}a_{12} &= a_{23}c_{32} & c_{32}a_{23} &= k_4c_{03} & c_{03}k_4 &= a_{01}a_{12}k_3 \\
k_4a_{01}a_{12}a_{23} &= k_5c_{03} & a_{01}a_{12}a_{23}k_4 &= c_{03}k_5 & & \\
k_5a_{01}a_{12}a_{23} &= a_{30}c_{03} & a_{01}a_{12}a_{23}k_5 &= c_{03}a_{30} & &
\end{array}$$

Definition 2.6. *For $r, a \in \mathbb{N}$ with $\mathrm{hcf}(r, a) = 1$ define the group $G = \frac{1}{r}(1, a)$ by*

$$G = \left\langle \zeta := \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle \leq GL(2, \mathbb{C}),$$

where ε is a primitive r^{th} root of unity.

Now consider the Jung-Hirzebruch continued fraction expansion of $\frac{r}{a}$, namely

$$\frac{r}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \frac{1}{\alpha_3 - \frac{1}{(\ldots)}}} := [\alpha_1, \ldots, \alpha_n]$$

with each $\alpha_i \geq 2$. The labelled Dynkin diagram of type A associated to this data is precisely the dual graph of the minimal resolution of $\mathbb{C}^2/\frac{1}{r}(1, a)$ [Rie77, Satz8].

Definition 2.7. *Define the reconstruction algebra $A_{r,a}$ associated to the group $G = \frac{1}{r}(1, a)$ to be the reconstruction algebra of type A associated to the data of the Jung-Hirzebruch continued fraction expansion of $\frac{r}{a}$.*

Note for the group $\frac{1}{r}(1,r-1)$, the reconstruction algebra $A_{r,r-1}$ is the reconstruction algebra of type A for the data $[\underbrace{2,\ldots,2}_{r-1}]$. Since $V_n=0$, $k_{V_n}=c_{10}$ and $l_{V_n}=1$ this is precisely the preprojective algebra of type $\widetilde{A}_{r-1}$.

Example 2.8. Since $\frac{7}{2}=[4,2]$ the reconstruction algebra $A_{7,2}$ associated to the group $\frac{1}{7}(1,2)$ is precisely the algebra in Example 2.4.

Example 2.9. After noticing that $\frac{40}{11}=[4,3,4]$ we see that the reconstruction algebra $A_{40,11}$ associated to the group $\frac{1}{40}(1,11)$ is precisely the algebra in Example 2.5.

The following lemma is important later for certain duality arguments; geometrically it says that the reconstruction algebra is independent of the direction we view the dual graph of the minimal resolution:

Lemma 2.10. *The reconstruction algebra of type A associated to the data $[\alpha_1,\ldots,\alpha_n]$ is the same as that associated to the data $[\alpha_n,\ldots,\alpha_1]$.*

Proof. If $n=1$ there is nothing to prove so assume $n\geq 2$. To avoid confusion write everything to do with the reconstruction algebra associated to $[\alpha_n,\ldots,\alpha_1]$ in typeface fonts eg $\mathtt{a}_{0n}$, $\mathtt{A}_{03}$, $\mathtt{c}_{12}$ $\mathtt{C}_{0n-1}$, $\mathtt{k}_1$, $\mathtt{u}_n$ etc. Flip the quiver vertex numbers by the operation $'$ which takes 0 to itself (ie $0'=0$), and reflects the other vertices in the natural line of symmetry (ie $1'=n$, $n'=1$ etc), then the explicit isomorphism between reconstruction algebras is given by

$$\begin{aligned} c_{ij} &\mapsto \mathtt{a}_{i'j'} \\ a_{ij} &\mapsto \mathtt{c}_{i'j'} \\ k_i &\mapsto \mathtt{k}_{n-i} \end{aligned}$$

Under this map $A_{ij}\mapsto \mathtt{C}_{i'j'}$ and $C_{ij}\mapsto \mathtt{A}_{i'j'}$, and furthermore the relations for the reconstruction algebra associated to $[\alpha_1,\ldots,\alpha_n]$ read backwards are precisely the relations for the reconstruction algebra associated to $[\alpha_n,\ldots,\alpha_1]$ read forwards. $\square$

Now $A_{r,a}$ is supposed to encode all information about the singularity $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ as well as the resolution, so since $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$ is determined by the continued fraction expansion of $\frac{r}{r-a}$ [Rie77, Satz8] it should be possible to read this directly from the quiver. Indeed this is true and to do it we must introduce some more notation.

Define $\sigma_1=1$ and inductively σ_s $(s\geq 1)$ to be the smallest vertex t with $t>\sigma_{s-1}$ and $\alpha_t>2$ (if it exists), else $\sigma_s=n$. Stop this process when we reach n. Thus we have

$$1=\sigma_1<\ldots<\sigma_z=n.$$

Note if all $\alpha_t=2$ this degenerates into $1=\sigma_1<\sigma_2=n$.

Lemma 2.11. *For the group $\frac{1}{r}(1,a)$ with notation as above*

$$\frac{r}{r-a}=[\underbrace{2,\ldots,2}_{u_{\sigma_1}-v_{\sigma_1}},(\sigma_2-\sigma_1+2),\underbrace{2,\ldots,2}_{u_{\sigma_2}-v_{\sigma_2}},(\sigma_3-\sigma_2+2),\underbrace{2,\ldots,2}_{u_{\sigma_3}-v_{\sigma_3}},\ldots,(\sigma_z-\sigma_{z-1}+2),\underbrace{2,\ldots,2}_{u_{\sigma_z}-v_{\sigma_z}}]$$

Proof. This is basically [Kid01, 1.2] translated into new language. $\square$

Example 2.12. By merely looking at the shape of $A_{40,11}$ in Example 2.2, by the above Lemma we can read off

$$\frac{40}{40-11}=[2,2,3,3,2,2]$$

Thus the shape of the reconstruction algebra $A_{r,a}$ determines the continued fraction expansion of $\frac{r}{r-a}$ which in turn determines the singularity $\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$. We shall see in the next section that in fact $Z(A_{r,a})=\mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$.

3. Special Cohen-Macaulay Modules

The reconstruction algebra is, by definition, constructed with prior knowledge of the minimal resolution. The aim of this section is to show that we could have defined it in a purely algebraic way by summing certain Cohen-Macaulay modules and looking at their endomorphism ring. More precisely in this section we shall show that the reconstruction algebra is isomorphic as a ring to the endomorphism ring of the sum of the special Cohen-Macaulay modules (together with the ring). In the process, we shall see that the relations for the reconstruction algebra arise naturally through a notion which we call a *web of paths*.

Keeping the notation from the last section, consider the group $G = \frac{1}{r}(1,a) = \langle \zeta \rangle$.

Definition 3.1. *For $0 \leq t \leq r-1$ define*

$$S_t = \{ f \in \mathbb{C}[x,y] : \zeta f = \varepsilon^t f \}.$$

These are precisely the non-isomorphic indecomposable maximal Cohen-Macaulay modules [Yos90, 10.10] over the Cohen-Macaulay singularity $X = \text{Spec } \mathbb{C}[x,y]^G$. Of these, only some are important:

Definition 3.2. [Wun88] *The module S_t is said to be special if $S_t \otimes \omega_X/\text{torsion}$ is Cohen-Macaulay, where ω_X is the canonical module of $X = \text{Spec } \mathbb{C}[x,y]^G$.*

There are in fact many equivalent characterisations of the special Cohen-Macaulay modules (see for example [Rie03, Thm 5]), some which refer to the minimal resolution and some that do not. For cyclic groups the combinatorics governing which CM modules are special is well understood.

Definition 3.3. *For $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$ define the i-series and j-series as follows:*

$$\begin{array}{lll} i_0 = r & i_1 = a & i_t = \alpha_{t-1} i_{t-1} - i_{t-2} \text{ for } 2 \leq t \leq n+1 \\ j_0 = 0 & j_1 = 1 & j_t = \alpha_{t-1} j_{t-1} - j_{t-2} \text{ for } 2 \leq t \leq n+1 \end{array}$$

It's easy to see

$$\begin{array}{ccccccccccc} i_0 = r & > & i_1 = a & > & i_2 & > & \ldots & > & i_n = 1 & > & i_{n+1} = 0 \\ j_0 = 0 & < & j_1 = 1 & < & j_2 = \alpha_1 & < & \ldots & < & j_n & < & j_{n+1} = r. \end{array}$$

It is the i-series which gives an easy combinatoric characterisation of the specials:

Theorem 3.4. [Wun87] *For $G = \frac{1}{r}(1,a)$ with $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$, the special Cohen-Macaulay modules are precisely those S_{i_p} for $0 \leq p \leq n$. Furthermore if $1 \leq p \leq n$ then S_{i_p} is minimally generated by x^{i_p} and y^{j_p}.*

For $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$ we now sum the specials and look at the endomorphism ring. Certainly there are the following maps between the specials:

$$\begin{tikzcd}
 & S_{i_{n-2}} \arrow[dl, bend right, "y^{j_{n-1}-j_{n-2}}"'] & \cdots\cdots & S_{i_4} \arrow[dr, "x^{i_3-i_4}"'] & \\
S_{i_{n-1}} \arrow[ur, "x^{i_{n-2}-i_{n-1}}"'] \arrow[d, bend right, "y^{j_n-j_{n-1}}"'] & & & & S_{i_3} \arrow[ul, bend right, "y^{j_4-j_3}"'] \arrow[d, "x^{i_2-i_3}"'] \\
S_1 \arrow[u, "x^{i_{n-1}-i_n}"'] \arrow[dr, bend right, "y^{j_{n+1}-j_n}=y^{r-j_n}"'] & & & & S_{i_2} \arrow[u, bend right, "y^{j_3-j_2}"'] \arrow[dl, "x^{i_1-i_2}"'] \\
 & S_0 \arrow[ul, "x^{i_n-i_{n+1}}=x"'] \arrow[rr, bend right, "y^{j_1-j_0}=y"'] & & S_{i_1} \arrow[ll, "x^{i_0-i_1}=x^{r-a}"'] \arrow[ur, bend right, "y^{j_2-j_1}"'] &
\end{tikzcd}$$

In general there will be more. If for any $1 \leq p \leq n$ it is true that $\alpha_p > 2$, then for each t such that $1 \leq t \leq \alpha_p - 2$, add an extra map from $S_{i_p} \to S_0$ labelled $x^{i_{p-1}-(t+1)i_p}y^{tj_p-j_{p-1}}$. Call the diagram complete with these extra arrows D; we now claim that D displays all the 'basic' maps between the specials, in that every other map between the specials must be a finite sum of compositions of these. This is important as it will show that the obvious map from $A_{r,a}$ to the endomorphism ring of the specials is surjective (see Theorem 3.23).

We firstly argue that for any $0 \leq p \leq n$ we can see any $f \in \operatorname{Hom}(S_{i_p}, S_{i_p}) \cong \mathbb{C}[x,y]^G$ as a finite sum of compositions of arrows in the above diagram forming a cycle at vertex p. We then argue that given any two specials S_{i_p}, S_{i_q} we can see any $f \in \operatorname{Hom}(S_{i_q}, S_{i_p}) \cong S_{i_p - i_q}$ as a finite sum of compositions of arrows in the above diagram from vertex p to vertex q. For the convenience of the reader we split this into a series of lemmas.

Lemma 3.5. *For any $0 \leq p \leq n$ we can view every $f \in \operatorname{Hom}(S_{i_p}, S_{i_p}) \cong \mathbb{C}[x,y]^G$ as a finite sum of compositions of arrows in the above diagram forming a cycle at vertex p.*

Proof. It is fairly clear that at every vertex $0, \ldots, n$ we have the following cycles

$$\begin{array}{rc}
 & x^r \\
 & x^{r-a}y \\
\text{if } \alpha_1 > 2 & \left\{ \begin{array}{c} x^{i_0-2i_1}y^{2j_1-j_0} \\ \vdots \\ x^{i_0-(\alpha_1-1)i_1}y^{(\alpha_1-1)j_1-j_0} \end{array} \right. \\
 & \vdots \\
\text{if } \alpha_n > 2 & \left\{ \begin{array}{c} x^{i_{n-1}-2i_n}y^{2j_n-j_{n-1}} \\ \vdots \\ x^{i_{n-1}-(\alpha_n-1)i_n}y^{(\alpha_n-1)j_n-j_{n-1}} \end{array} \right. \\
 & y^r
\end{array}$$

But by Lemma 2.11 and [Rie77, Satz8] these form the minimal generating set for $\mathbb{C}[x,y]^G$ and so certainly f can be viewed as a finite sum of cycles at the vertex p. $\square$

Lemma 3.6. *For any $0 \leq p \leq n$, any map $S_0 \to S_{i_p}$ can be seen in the diagram.*

Proof. The $p = 0$ case is Lemma 3.5 so assume $p > 0$. Then $\operatorname{Hom}(S_0, S_{i_p}) \cong S_{i_p}$ which by Theorem 3.4 is generated as a module by x^{i_p} and y^{j_p}. Clearly both of these generators are paths in the diagram and so since cycles at vertex p are all of $\mathbb{C}[x,y]^G$ we are done. $\square$

Thus it remains to prove the following 2 statements:

(i) for any $0 \leq q < p \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in the diagram.
(ii) for any $0 < p < q \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in the diagram.

We shall see that we need only prove (i), then appealing to duality gives (ii) for free. In what follows we refer to the vertex S_{i_t} as the t^{th} vertex.

Lemma 3.7. *For $0 \leq q < p \leq n$, if $x^{z_1}y^{z_2} \in S_{i_q - i_p}$ with $0 \leq z_1, z_2 \leq r-1$, then $x^{z_1}y^{z_2}$ factors as either*
(i) $(x^{i_{p-1}-i_p})A$ *for some* $A \in S_{i_q - i_{p-1}}$
(ii) $(x^{i_{p-1}-(t+1)i_p}y^{tj_p-j_{p-1}})B$ *for some* $B \in S_{i_q}$ *and some* $1 \leq t \leq \alpha_p - 2$
(iii) $(y^{j_{p+1}-j_p})C$ *for some* $C \in S_{i_q - i_{p+1}}$.

Proof. The case $p = n$ is trivial, so assume $p < n$. Clearly if $z_1 \geq i_{p-1} - i_p$ then we're in (i) so we can assume that $0 \leq z_1 < i_{p-1} - i_p$. Consider the invariant $x^{z_1}y^{z_2+(j_p-j_q)}$. Since we can see all invariants at every vertex, consider this as a path in D at the p^{th} vertex. It must leave the vertex, and the hypothesis on z_1 means that it can't leave through the $x^{i_{p-1}-i_p}$ map.

Case 1: $\alpha_p = 2$ Then it must leave through the $y^{j_{p+1}-j_p}$ map to vertex $p+1$, i.e.

$$x^{z_1}y^{z_2+(j_p-j_q)} = y^{j_{p+1}-j_p}M$$

for some path M from vertex $p+1$ to q. Now from vertex $p+1$ the path M has to reach vertex p again. But this can only be reached in two ways, via the map $x^{i_{p-1}-i_p} = x^{i_p-i_{p+1}}$ from vertex $p+1$ to p, or via the map $y^{j_p-j_{p-1}}$ from vertex $p-1$ to p. The hypothesis on the x forces the later, so in particular the path factors through the 0 vertex. It may be true that there are cycles in the path that occur after the 0th vertex however since we have all cycles at all vertices we may move these cycles to the 0th vertex and hence assume that the path M finishes as the composition

$$y^{j_1-j_0}y^{j_2-j_1}\dots y^{j_p-j_{p-1}} = y^{j_p}$$

Hence we may write

$$x^{z_1}y^{z_2+(j_p-j_q)} = y^{j_{p+1}-j_p}Ay^{j_p}$$

for some path $A : S_{i_{p+1}} \to S_0$. But since $j_p \geq j_p - j_q$ we can cancel $y^{j_p-j_q}$ from both sides and write

$$x^{z_1}y^{z_2} = y^{j_{p+1}-j_p}A'$$

for some monomial A'. Necessarily $A' \in S_{i_q-i_{p+1}}$ and so we have the desired factorisation as in (iii).

Case 2: $\alpha_p > 2$. For notational ease denote the extra arrows leaving vertex p by $k_t = x^{i_{p-1}-(t+1)i_p}y^{tj_p-j_{p-1}}$. Now $x^{z_1}y^{z_2+(j_p-j_q)}$ must leave vertex p through the $y^{j_{p+1}-j_p}$ map to vertex $p+1$, or through one of the extra k_t. We argue case by case:

Suppose first that $x^{z_1}y^{z_2+(j_p-j_q)}$ leaves through the $y^{j_{p+1}-j_p}$ map to vertex $p+1$, i.e.

$$x^{z_1}y^{z_2+(j_p-j_q)} = y^{j_{p+1}-j_p}M$$

for some path M from vertex $p+1$ to p. If M leaves vertex $p+1$ through the $x^{i_p-i_{p+1}}$ map we are done since then

$$x^{z_1}y^{z_2+(j_p-j_q)} = x^{i_p-i_{p+1}}y^{j_{p+1}-j_p}M_1 = k_{\alpha_p-2}y^{j_p}M_1$$

for some monomial M_1 and so since $j_p \geq j_p - j_q$ we may cancel and write $x^{z_1}y^{z_2} = k_{\alpha_p-2}M_1'$ for some monomial M_1' which necessarily belongs to S_{i_q}; this is a factorisation as in (ii). Hence we can assume that M leaves vertex $p+1$ via another path. Since $p+1 \leq n$ each of these paths has y component greater than or equal to $y^{j_{p+1}-j_p}$ and so we may write

$$x^{z_1}y^{z_2+(j_p-j_q)} = y^{2(j_{p+1}-j_p)}M_2$$

for some monomial M_2. But now $j_{p+1} - j_p > j_p - j_q$ so we may cancel and write $x^{z_1}y^{z_2} = y^{j_{p+1}-j_p}M_2'$ for some monomial M_2' which necessarily belongs to $S_{i_q-i_{p+1}}$; this is a factorisation as in (iii).

Now suppose that $x^{z_1}y^{z_2+(j_p-j_q)}$ factors through one of the extra arrows k_t out of vertex p. Thus $x^{z_1}y^{z_2+(j_p-j_q)} = k_tB$ for some $1 \leq t \leq \alpha_p - 2$ and some path B from 0 to p. By Lemma 3.6 there are 2 possibilities for B: either $B = x^{i_p}B_1$ or $B = y^{j_p}B_2$ for some invariants B_1 and B_2. We split the remainder of the proof into cases depending on the value of t:

If $t = 1$ then $x^{z_1}y^{z_2+(j_p-j_q)}$ is either $k_1x^{i_p}B_1 = x^{i_{p-1}-i_p}y^{j_p-j_{p-1}}B_1$ which is impossible by the assumption on z_1, or it's equal to $k_1y^{j_p}B_2$. But now $j_p - j_q \leq j_p$ and so after cancelling we may write $x^{z_1}y^{z_2} = k_1B_2'$ for some monomial B_2' which necessarily belongs to S_{i_q}; this gives a factorization as in (ii).

The above argument takes care of $t = 1$ and so we are done if $\alpha_p = 3$. Hence the final case to consider is when $\alpha_p > 3$ and t is such that $1 < t \leq \alpha_p - 2$. Here $x^{z_1}y^{z_2+(j_p-j_q)}$ is either

$$k_tx^{i_p}B_1 = k_{t-1}y^{j_p}B_1 \quad \text{or} \quad k_ty^{j_p}B_2.$$

Again $j_p - j_q \leq j_p$ and so after cancelling we may write $x^{z_1}y^{z_2}$ as either

$$k_{t-1}B_1' \quad \text{or} \quad k_tB_2'$$

for some monomials B_2', B_2' which necessarily belong to S_{i_q}. This gives the required factorizations as in (ii), and completes the proof. □

The next two results are simple inductive arguments based on the previous lemma.

Corollary 3.8. *For any $0 \leq q < n$, every map $S_{i_n} \to S_{i_q}$ can be seen in the diagram.*

Proof. Let $x^{z_1}y^{z_2} \in S_{i_q - i_n} = S_{i_q - 1}$ then by Lemma 3.5 we can remove cycles and so assume $0 \leq z_1, z_2 \leq r-1$. By Lemma 3.7 we know $x^{z_1}y^{z_2}$ either factors
(i) through vertex $n-1$ as $(x^{i_{n-1}-i_n})A$ for some map $A : S_{i_{n-1}} \to S_{i_q}$
(ii) through vertex 0 as $(x^{i_{n-1}-(t+1)i_n}y^{tj_n - j_{n-1}})B$ for some $B : S_0 \to S_{i_q}$ and some $1 \leq t \leq \alpha_n - 2$
(iii) through vertex 0 as $(y^{j_{n+1}-j_n})C$ for some $C \in S_{i_q - i_{p+1}} = S_{i_q} = \mathrm{Hom}(S_0, S_{i_q})$.
If we are in cases (ii) or (iii) then we're done by Lemma 3.6 since we can see both B and C in the diagram. Hence assume case (i). If $q = n-1$ then we are done since by Lemma 3.5 we can see A in the diagram. Hence we can assume that we are in case (i) with $q < n-1$. But now by Lemma 3.7 A either factors
(i) through vertex $n-2$ as $(x^{i_{n-2}-i_{n-1}})A'$ for some map $A' : S_{i_{n-2}} \to S_{i_q}$
(ii) through vertex 0 as $(x^{i_{n-2}-(t+1)i_{n-1}}y^{tj_{n-1}-j_{n-2}})B'$ for some $B' : S_0 \to S_{i_q}$ and some $1 \leq t \leq \alpha_{n-1} - 2$
(iii) through vertex n as $(y^{j_n - j_{n-1}})C'$ for some $C' : S_n \to S_q$.
Since we removed cycles from $x^{z_1}y^{z_2}$ at the beginning we can't be in case (iii). If in case (ii) then we're again done by Lemma 3.6, so can again assume in case (i). If $q = n-2$ then we again we are done, so we can suppose in case (i) with $q < n-2$. Proceed inductively; since $0 \leq q$ this process must stop. □

Corollary 3.9. *For any $0 \leq q < p \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in the diagram.*

Proof. Fix q. This is now just a simple induction argument: if $p = n$ then the result is true by Corollary 3.8, so let $p < n$ and assume the result is true for larger p.

Let $x^{z_1}y^{z_2} \in S_{i_q - i_p}$ then by Lemma 3.5 we can remove cycles and so assume $0 \leq z_1, z_2 \leq r-1$. By Lemma 3.7 we know $x^{z_1}y^{z_2}$ either factors
(i) through vertex $p-1$ as $(x^{i_{p-1}-i_p})A$ for some map $A : S_{i_{p-1}} \to S_{i_q}$
(ii) through vertex 0 as $(x^{i_{p-1}-(t+1)i_p}y^{tj_p - j_{p-1}})B$ for some $B : S_0 \to S_{i_q}$ and some $1 \leq t \leq \alpha_p - 2$
(iii) through vertex $p+1$ as $(y^{j_{p+1}-j_p})C$ for some $C : S_{i_{p+1}} \to S_{i_q}$.
If we are in case (iii) then by inductive hypothesis we can see C in the diagram and so we are done; if in case (ii) then by Lemma 3.6 we are also finished. Hence we can assume we are in case (i). If $q = p-1$ then we are done by Lemma 3.5 as we can see A in the diagram, hence we can assume $q < p-1$. Thus the result follows by an identical argument as in Corollary 3.8 above - we've removed cycles and so A can't factor through S_{i_p}. □

To prove the corresponding statement in the opposite direction we appeal to duality. More precisely the singularity defined by $\frac{1}{r}(1,a)$ with $\frac{r}{a} = [\alpha_1, \ldots, \alpha_n]$ is isomorphic to the singularity $\frac{1}{r}(1,b)$ with $\frac{r}{b} = [\alpha_n, \ldots, \alpha_1]$ (note $b = j_n$); the isomorphism is given by swapping the x and y's. To avoid confusion refer to everything for the singularity $\frac{1}{r}(1,b)$ in typeface font, eg Cohen-Macaulay modules $\mathtt{S}_t$, i-series by $\mathtt{i}$, diagram $\mathtt{D}$ etc. The explicit isomorphism is given by

$$\begin{array}{rcl} S_0 & \to & \mathtt{S}_0 \\ x & \mapsto & \mathtt{y} \\ y & \mapsto & \mathtt{x} \end{array}$$

As in Lemma 2.10 flip the quiver vertex numbers by the operation $'$ which takes 0 to itself (ie $0' = 0$), and reflects the other vertices in the natural line of symmetry (ie $1' = n$, $n' = 1$ etc). Now for all $1 \leq p \leq n$ we have $i_p = \mathtt{j}_{p'}$ and $j_p = \mathtt{i}_{p'}$ thus

$$S_{i_p} = (x^{i_p}, y^{j_p})S_0 \cong (\mathtt{y}^{\mathtt{j}_{p'}}, \mathtt{x}^{\mathtt{i}_{p'}})\mathtt{S}_0 = \mathtt{S}_{\mathtt{i}_{p'}}.$$

Corollary 3.10. *For the singularity $\frac{1}{r}(1,a)$, for any $0 \leq p < q \leq n$, every map $S_{i_p} \to S_{i_q}$ can be seen in the diagram D.*

Proof. Under the duality above $x^{z_1}y^{z_2} : S_{i_p} \to S_{i_q}$ corresponds to $\mathtt{y}^{z_1}\mathtt{x}^{z_2} : \mathtt{S}_{\mathtt{i}_{p'}} \to \mathtt{S}_{\mathtt{i}_{q'}}$. But since $q' < p'$ we can by Corollary 3.9 view this in the diagram $\mathtt{D}$ as the composition of monomials whose powers are in terms of $\mathtt{i}$'s, $\mathtt{j}$'s and α_t's. Under the duality isomorphisms we can view this as a path in the diagram D. $\square$

Summarizing Lemma 3.5, Corollary 3.9 and Corollary 3.10 we have:

Proposition 3.11. *For any $\frac{1}{r}(1,a)$, for any $0 \leq p, q \leq n$, we can see any map $S_{i_p} \to S_{i_q}$ in the diagram D.*

This may seem abstract but in reality it is extremely useful if we actually want to compute some examples. One way to compute the endomorphism ring of the specials is to take the McKay quiver and corner (ie ignore a vertex and compose maps that pass through that vertex) the non-special vertices. Of course the larger the group the longer this computation; for the example $\frac{1}{40}(1,11)$ there are forty vertices in the McKay quiver and we need to corner thirty-six of them. Given any example $\frac{1}{r}(1,a)$, Proposition 3.11 saves us this long computation since the algorithm to produce the necessary diagram involves only the continued fraction expansion of $\frac{r}{a}$ and the associated i and j series, all of which are extremely quick to calculate.

Example 3.12. For $\frac{1}{40}(1,11)$, $\frac{40}{11} = [4,3,4]$ so the i and j-series are

$$\begin{array}{ccccccccc} i_0 = 40 & > & i_1 = 11 & > & i_2 = 4 & > & i_3 = 1 & > & i_4 = 0 \\ j_0 = 0 & < & j_1 = 1 & < & j_2 = 4 & < & j_3 = 11 & < & j_4 = 40. \end{array}$$

By Proposition 3.11 the endomorphism ring of the specials is

$$\begin{tikzcd} S_1 \arrow[r, "x^3"] \arrow[d, bend right=60, "y^{29}"'] \arrow[d, bend right=40, "xy^{18}"] \arrow[d, bend right=20, "x^2y^7"] & S_4 \arrow[l, bend right, "y^7"'] \arrow[dl, "x^3y^3"] \arrow[d, "x^7"] \\ S_0 \arrow[u, "x"] \arrow[r, bend right, "y"'] & S_{11} \arrow[l, shift right=2, "x^7y^2"'] \arrow[l, "x^{18}y"] \arrow[l, shift left=2, "x^{29}"] \arrow[u, bend right, "y^3"'] \end{tikzcd}$$

Notice the correspondence with Example 2.2.

Example 3.13. For $\frac{1}{7}(1,2)$, $\frac{7}{2} = [4,2]$ so the i and j-series are

$$\begin{array}{ccccccc} i_0 = 7 & > & i_1 = 2 & > & i_2 = 1 & > & i_3 = 0 \\ j_0 = 0 & < & j_1 = 1 & < & j_2 = 4 & < & j_3 = 7. \end{array}$$

By Proposition 3.11

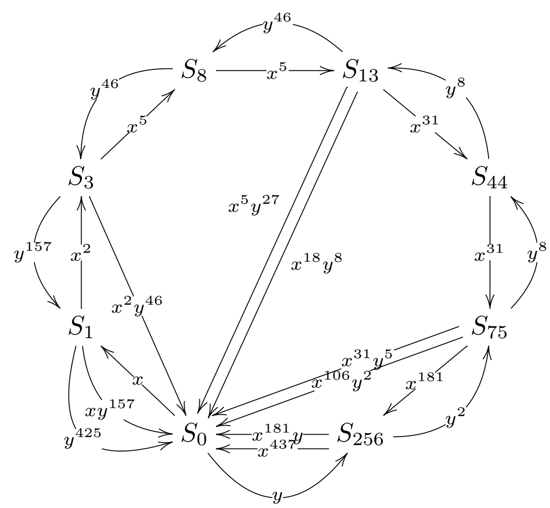
$$\mathrm{End}(S_0 \oplus S_1 \oplus S_2) = \begin{tikzcd} & S_1 \arrow[dl, bend right, "y^3"'] \arrow[dr, "x"] & \\ S_0 \arrow[ur, "x"] \arrow[rr, bend right, "y"'] & & S_2 \arrow[ll, shift right=2, "xy^2"'] \arrow[ll, "x^3y"] \arrow[ll, shift left=2, "x^5"] \arrow[ul, bend right, "y^3"'] \end{tikzcd}$$

Notice the correspondence with Example 2.1.

Example 3.14. For the group $\frac{1}{693}(1,256)$, $\frac{693}{256}=[3,4,2,4,2,3,3]$ so the i and j series are

	0	1	2	3	4	5	6	7	8
i	693	256	75	44	13	8	3	1	0
j	0	1	3	11	19	65	111	268	693.

and further the endomorphism ring of the specials is



In the above we have described the endomorphism ring of the special Cohen-Macaulay modules as a quiver, subject to the relations that x and y commute wherever that makes sense. We now want to show that the relations from the reconstruction algebra are all the relations that are needed.

To achieve this double index the arrows in $A_{r,a}$ as follows:

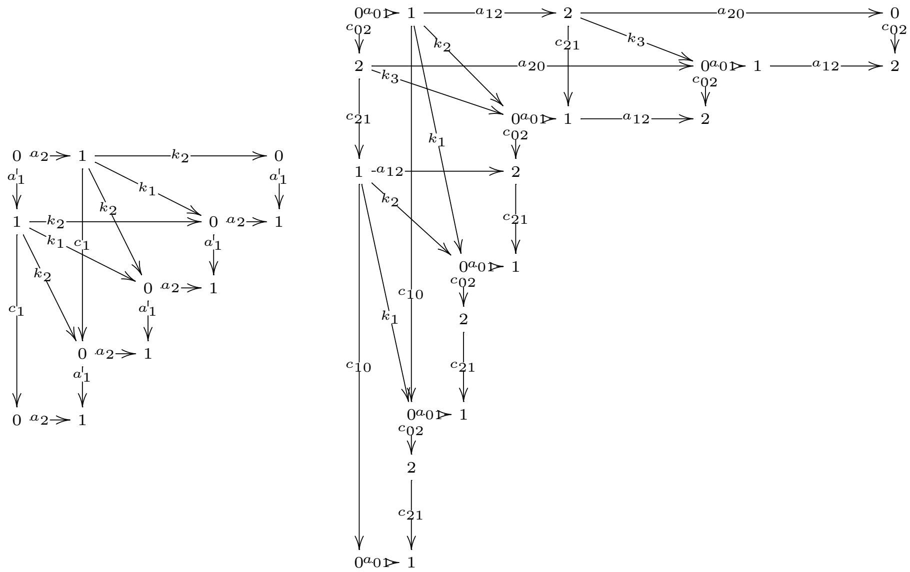
arrow	double index
c_{0n}	$(1,0)$
c_{tt-1}	$(i_{t-1}-i_t,0)$
a_{n0}	$(0,r-j_n)$
a_{tt+1}	$(0,j_{t+1}-j_t)$
k_s	$(i_{l_s-1}-((s-V_{l_s})+1)i_{l_s},(s-V_{l_s})j_{l_s}-j_{l_s-1})$

It is easy to see that the two terms in any relation for $A_{r,a}$ have the same double index and so the double index can be extended to all paths in $A_{r,a}$. We shall now show that if there exists a path of double index (z_1,z_2) leaving a vertex t in $A_{r,a}$ then the path is necessarily unique.

Definition 3.15. *For a given vertex t in $A_{r,a}$ define the web of paths leaving t as follows: place t in the $(0,0)$ position of a 2-dimensional grid, and for each arrow leaving t draw a line from $(0,0)$ to the double index of that arrow. Mark the end of this line by the head of the arrow. Continue in this way for all the heads of the arrows.*

Like most definitions, this is best understood after consulting some examples. In the following two examples the web should be extended forever in the obvious direction; for practical purposes we draw only the start of the picture.

Example 3.16. The web of paths from vertex 0 in $A_{4,1}$ and $A_{11,3}$ begins respectively:



We call the points in the web of paths that lie in the set $\{(w,0) : 0 \leq w < n\}$ the left rail and similarly those that lie in the set $\{(0,w) : 0 \leq w < n\}$ are called the top rail.

Definition 3.17. *Draw the left rail and the top rail in the web of paths leaving vertex* 0*, and draw in every arrow leaving these vertices. Join the ends of these by using only vertical and horizontal paths, and call the resulting diagram* F*.*

The fact that this can always be done is due to the grading we put on $A_{r,a}$, together with simple combinatorics with continued fractions. The examples above show F for $A_{4,1}$ and $A_{11,3}$.

It is fairly clear that F generates the web of paths leaving 0 in the sense that all paths can be obtained by gluing on extra copies of F to the existing copy. The copies of F glue together seamlessly due to the symmetry and repetition inside F. No new basic squares are created in this process since the boundary of F consists entirely of straight lines and also F contains (by definition) all paths from the rails so there can be no paths that leap over the boundary to create new squares.

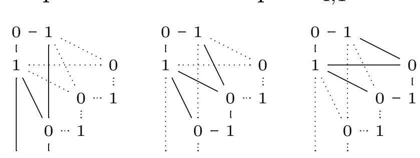
Since F generates the web of paths leaving 0 it is clear (since F can) that the web of paths can be subdivided into small 'squares'; we call these *basic squares*.

Definition 3.18. *A square is a pair of paths* $(p_1 \dots p_s, q_1 \dots q_t)$ *with* $tail(p_1) = tail(q_1)$ *and* $head(p_s) = head(q_t)$*. A square in* F *is called basic if* $p_i \notin \{q_1, \dots, q_t\}$ *for all* $1 \leq i \leq s$ *and* $q_j \notin \{p_1, \dots, p_s\}$ *for all* $1 \leq j \leq t$*.*

By the definition and structure of F it is clear that if all basic squares in F commute then all squares in F commute. Since F generates the web of paths this means all squares commute (since they are made from squares in F), giving us the required uniqueness of path.

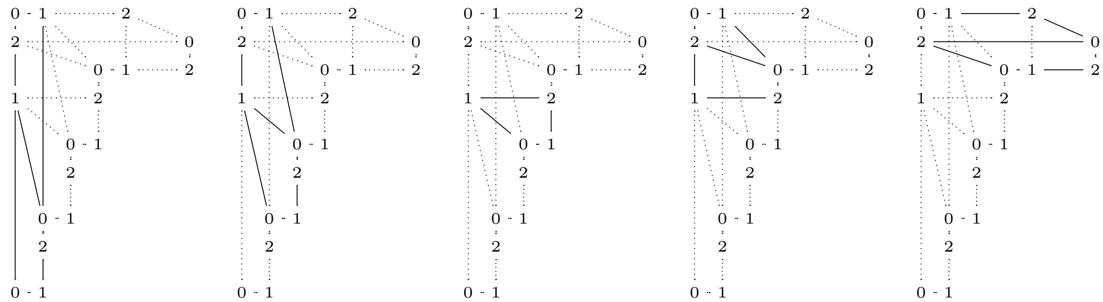
Because of the symmetry in F there are in fact repetitions of the basic squares inside F - more precisely the basic squares starting at the 1 on the top rail are the same as those starting at 1 on the left rail, etc. Thus by the symmetry in F it is clear that all the basic squares leaving the left rail are *all* the basic squares in F.

Example 3.19. The 6 basic squares in the example $A_{4,1}$ above are



which are precisely the relations. Note that these prove the paths $a_2k_1a_1$ and $a_1c_1k_2$ coincide since that square can be subdivided into 2 basic squares, both of which commute.

Example 3.20. The 9 basic squares in the example $A_{11,3}$ above are



The first three diagrams are the five Step 1 relations, the last two diagrams are the four Step 2 relations.

Note for example that in the web of paths for $A_{4,1}$ above the paths $a_2k_1a_1$ and $a_1c_1k_2$ coincide since the square can be subdivided into 2 basic squares, both of which commute.

Lemma 3.21. *For any double index (z_1, z_2) either there is precisely one path out of vertex 0 with that double index, or there are none.*

Proof. By the above we just need to prove that all the basic squares in F out of the left rail commute. This is just a bookkeeping exercise:
Case 1: $n = 1$. This is an easy extension of the $A_{4,1}$ example above.
Case 2: $n > 1$. We work through the basic squares leaving 1 (which we'll see, together with some basic squares leaving 0, correspond to the Step 1 relations) and then work upwards: if $\alpha_1 = 2$ then the only basic square leaving 1 is $c_{10}a_{01} = a_{12}c_{21}$, so we may assume $\alpha_1 > 2$. Then, as in Example 3.20 we get $k_sa_{01} = k_{s+1}C_{01}$ and above it $a_{01}k_s = C_{01}k_{s+1}$ for all $0 \leq s < u_1$, then end with $k_{u_1}a_{01} = a_{12}c_{21}$. Thus all basic squares leaving 1 (and the corresponding ones leaving 0) on the left rail commute.
Now for basic squares leaving t on the left rail with $1 < t < n$ (if such t exist): if $\alpha_t = 2$ then the only basic square is $c_{tt-1}a_{t-1t} = a_{tt+1}c_{t+1t}$ so we may assume that $\alpha_t > 2$. Certainly we have the basic square $c_{tt-1}a_{t-1t} = k_{v_t}C_{0t}$ and above it $C_{0t}k_{v_t} = A_{0l_{V_t}}k_{V_t}$. If $\alpha_t > 3$ we also have the basic squares $k_sA_{0t} = k_{s+1}C_{0t}$ and above it $A_{0t}k_s = C_{0t}k_{s+1}$ for all $v_t \leq s < u_t$. The final basic square out of t is $k_{u_t}A_{0t} = a_{tt+1}c_{t+1t}$.
For the basic squares leaving n on the left rail: if $\alpha_n = 2$ then the only basic square is $c_{nn-1}a_{n-1n} = a_{n0}c_{0n}$ and above it $c_{0n}a_{n0} = A_{0l_{V_n}}k_{V_n}$. Hence assume $\alpha_n > 2$. Then $c_{nn-1}a_{n-1n} = k_{v_n}c_{0n}$ and above it $c_{0n}k_{v_n} = A_{0l_{V_n}}k_{V_n}$. The only basic squares remaining are $k_sA_{0n} = k_{s+1}c_{0n}$ and above it $A_{0n}k_s = c_{0n}k_{s+1}$ for all $v_n \leq s < u_n$ (recall $k_{u_n} = a_{n0}$). $\square$

Corollary 3.22. *For any double index (z_1, z_2) and any vertex t, either there is precisely one path out of vertex i with that double index, or there are none.*

Proof. To obtain the web of paths of vertex n delete the top row in the web of paths of vertex 0 and decrease the first index in every double index by 1. All squares in this web of paths commute because they do in the web of paths for vertex 0. For vertex $n-1$ delete all the rows above the $n-1$ on the left rail, and decrease the double indices accordingly. Again all squares in this web of paths commute since they do in the web of paths for vertex 0. Continue in this fashion. $\square$

We now reach the main theorem which shows that the algebraically-constructed ring (the endomorphism ring of the specials) is isomorphic to the geometrically-constructed ring (the reconstruction algebra). For a third construction of the same non-commutative ring, see Section 5.

Theorem 3.23. *For $G = \frac{1}{r}(1,a)$, let $T_{r,a} = \oplus_{p=1}^{n+1} S_{i_p}$. Then*
$$A_{r,a} \cong \mathrm{End}_{\mathbb{C}[x,y]^G}(T_{r,a}).$$

Proof. By using the above quiver description of $\mathrm{End}(T_{r,a})$ as the diagram D with the relations that x and y commute whenever that makes sense, there is an obvious correspondence with $A_{r,a}$. More precisely define a map $\phi : A_{r,a} \to \mathrm{End}(T_{r,a})$ by

$$\begin{array}{rcl} c_{0n} & \mapsto & x^{i_n - i_{n+1}} = x \\ c_{tt-1} & \mapsto & x^{i_{t-1}-i_t} \\ a_{n0} & \mapsto & y^{j_{n+1}-j_n} = y^{r-j_n} \\ a_{tt+1} & \mapsto & y^{j_{t+1}-j_t} \\ k_s & \mapsto & x^{i_{l_s-1}-((s-V_{l_s})+1)i_{l_s}} y^{(s-V_{l_s})j_{l_s}-j_{l_s-1}} \end{array}$$

where recall $V_1 := 0$. It is straightforward that ϕ is well defined since the double index of any relation corresponds to the double index (z_1, z_2) from the cycle $x^{z_1}y^{z_2}$ in the endomorphism ring that it represents.

By Proposition 3.11 the map is surjective; the content in the theorem is that it also injective, ie there are no more relations. But this is just Corollary 3.22. □

If $a = r-1$ then the group $\frac{1}{r}(1, r-1)$ is inside $SL(2,\mathbb{C})$, all Cohen-Macaulay modules are special and $T_{r,r-1} = \mathbb{C}[x,y]$ so this theorem reduces to the well known
$$A_{r,r-1} = \text{pre-projective algebra} \cong \mathrm{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y]) \cong \mathbb{C}[x,y]\#G,$$
which is certainly the non-commutative ring that dictates the process of desingularisation (cf [KV00]).

4. Moduli Space of Representations and the Minimal Resolution

The minimal resolutions of cyclic quotient singularities are well understood by a construction of Fujiki (see for example [Wun87, 2.7]); more recently there is an easier algorithm using toric geometry techniques [Rei97] which coincides with the G-Hilb description ([Kid01], [Ish02]).

In this section we prove that for any group $G = \frac{1}{r}(1,a)$ we can obtain the minimal resolution of the singularity $\mathbb{C}^2/G$ from the moduli space of the reconstruction algebra $A_{r,a}$, thus giving yet another description of the minimal resolution. This may not be entirely surprising (by construction!), but it is important since by Theorem 3.23 we could have defined $A_{r,a}$ without prior knowledge of the minimal resolution.

For a summary of moduli space techniques we refer the reader to [Kin94], [Kin97]. For $G = \frac{1}{r}(1,1)$ (ie the reconstruction algebra with the $n=1$ relations) everything is trivial so we assume $n \geq 2$. With respect to the ordering of the vertices as in Section 2, fix for the rest of this paper the dimension vector $\alpha = (1,1,\ldots,1)$ and fix the generic stability condition $\theta = (-n,1,\ldots,1)$. The point is that when considering representations of this dimension vector the maps are just scalars so the relations reduce in complexity. As we shall see the stability condition is chosen to be 'blind' to the arrows $k_1, \ldots k_{\sum(\alpha_1-2)}$ and so we have a open covering of the moduli space by the same number of opens as in the preprojective case (ie $n+1$ open sets). It is the relations that ensure each of the opens is still $\mathbb{C}^2$, and standard geometric arguments give minimality. For toric geometers, there is the following identification between the toric description of the minimal resolution and the moduli space description:

$$\begin{array}{rcccl} W_0 : & C_{01} \neq 0 & \longrightarrow & U_0 = \mathbb{C}\left[x^r, \frac{y}{x^a}\right] & \text{as } (C_{00}\frac{a_{01}}{C_{01}}) \\ & & \vdots & & \\ \begin{array}{c} W_t \\ \text{for } 1 \leq t \leq n-1 \end{array} : & C_{0t+1} \neq 0, A_{0t} \neq 0 & \longrightarrow & U_t = \mathbb{C}\left[\frac{x^{i_t}}{y^{j_t}}, \frac{y^{j_{t+1}}}{x^{i_{t+1}}}\right] & \text{as } (\frac{C_{0t}}{A_{0t}}, \frac{A_{0t+1}}{C_{0t+1}}) \\ & & \vdots & & \\ W_n : & A_{0n} \neq 0 & \longrightarrow & U_n = \mathbb{C}\left[\frac{x}{y^{j_n}}, y^r\right] & \text{as } (\frac{c_{0n}}{A_{0n}}, A_{00}) \end{array}$$

Lemma 4.1. *For the group $G = \frac{1}{r}(1, a)$, with notation as above $\{W_t : 0 \leq t \leq n\}$ forms an open cover of the moduli space* $\mathrm{Rep}(A_{r,a}, \alpha)/\!\!/_\theta \mathrm{GL}$.

Proof. Suppose $M \in \mathrm{Rep}(A_{r,a}, \alpha)$ is θ-stable. It is clear from the stability condition that we need, for every vertex $i \neq 0$, a non-zero path from vertex 0 to vertex i. Now if $a_{01} = 0$ then to get to vertex 1 clearly we need $C_{01} \neq 0$ and so M is in W_0. Hence we can assume $a_{01} \neq 0$. If $a_{12} = 0$ then to get to vertex 2 we need $C_{02} \neq 0$ and so M is in W_1. Continuing in this fashion it is obvious that $\{W_t : 0 \leq t \leq n\}$ forms an open cover of the moduli space. □

The next lemma is trivial in any given example, but the general proof is a little ackward to write down:

Lemma 4.2. (i) *Each representation in W_0 is determined by $(c_{10}, a_{01}) \in \mathbb{C}^2$.*
(ii) *For every $1 \leq t \leq n-1$, each representation in W_t is determined by $(c_{t+1t}, a_{tt+1}) \in \mathbb{C}^2$.*
(iii) *Each representation in W_n is determined by $(a_{n0}, c_{0n}) \in \mathbb{C}^2$.*
Thus every open set in the cover is just $\mathbb{C}^2$.

Proof. (i) We can set $c_{0n} = c_{nn-1} = \ldots = c_{21} = 1$. We proceed anticlockwise round the vertices of the quiver (starting at the 0th vertex), showing at each stage that all arrows leaving the vertex are determined by c_{10} and a_{01}.
Vertex 0: trivial as the only arrows leaving are $c_{0n} = 1$ and a_{01}.
Vertex 1: If $\alpha_1 = 2$ then the only two arrows leaving are a_{12} and c_{10}. The Step 1 relations give $a_{12} = c_{10}a_{01}$. Thus we may assume that $\alpha_1 > 2$ so we have $c_{10} = k_0, k_1, \ldots, k_{u_1}, a_{12}$ leaving the vertex. But now the Step 1 relations give

$$\begin{array}{rcl} c_{10}a_{01} &=& k_1 \\ k_1a_{01} &=& k_2 \\ &\vdots& \\ k_{u_1-1}a_{01} &=& k_{u_1} \\ k_{u_i}a_{01} &=& a_{12} \end{array}$$

so it is clear that $k_1, \ldots, k_{u_1}, a_{12}$ can be expressed in terms of c_{10} and a_{01}.
Vertex s for $1 < s < n$: If $\alpha_s = 2$ then only arrows leaving are $c_{ss-1} = 1$ and a_{ss+1}. The Step s relations give $a_{ss+1} = a_{s-1s}$ and by work at previous vertices we know that a_{s-1s} is determined by c_{10} and a_{01}; hence so is a_{s+1s}. Thus we may assume $\alpha_s > 2$ and so the arrows leaving vertex s are $k_{v_s}, \ldots, k_{u_s}, c_{ss-1} = 1, a_{ss+1}$. But by the Step s relations we know

$$\begin{array}{rcl} k_{v_s} &=& a_{s-1s} \\ k_{v_s+1} &=& k_{v_s}A_{0s} \\ &\vdots& \\ k_{u_s} &=& k_{u_s-1}A_{0s} \\ a_{ss+1} &=& k_{u_s}A_{0s} \end{array}$$

which, by work at the previous vertices, can all be expressed in terms of c_{10} and a_{01}.
Vertex n: If $\alpha_n = 2$ then again everything is trivial and so we may assume $\alpha_n > 2$ in which case the arrows $k_{v_n}, \ldots, k_{u_n} = a_{n0}, c_{nn-1} = 1$ leave vertex n. The step n relations give

$$\begin{array}{rcl} k_{v_n} &=& a_{n-1n} \\ k_{v_n+1} &=& k_{v_n}A_{0n} \\ &\vdots& \\ k_{u_n} &=& k_{u_s-1}A_{0s} \end{array}$$

which again by work at the other vertices can be expressed in terms of c_{10} and a_{01}.
(iii) Follows immediately by Lemma 2.10.
(ii) We can set $c_{0n} = \ldots = c_{t+2\,t+1} = 1 = a_{01} = \ldots = a_{t-1t}$. As above we show that the arrows leaving every vertex are determined by c_{t+1t} and a_{tt+1}, but we now work anticlockwise from vertex $t+1$ to vertex 0, then work clockwise from vertex t to vertex 1:
Vertex $t+1$: If $\alpha_{t+1} = 2$ then the only arrows leaving are c_{t+1t} and a_{t+1t+2}. The relations give $a_{t+1t+2} = c_{t+1t}a_{tt+1}$. Hence we may assume $\alpha_{t+1} > 2$ and so the arrows leaving vertex

$t+1$ are $k_{v_{t+1}},\ldots,k_{u_{t+1}},c_{t+1t},a_{t+1t+2}$. The Step $t+1$ relations give

$$\begin{aligned} k_{v_{t+1}} &= c_{t+1t}a_{tt+1}\\ k_{v_{t+1}+1} &= k_{v_{t+1}}A_{0t+1}=k_{v_{t+1}}a_{tt+1}\\ &\vdots\\ k_{u_{t+1}} &= k_{u_{t+1}-1}A_{0t+1}=k_{u_{t+1}-1}a_{tt+1}\\ a_{t+1t+2} &= k_{u_{t+1}}A_{0t+1}=k_{u_{t+1}}a_{tt+1}\end{aligned}$$

which therefore can all be expressed in terms of c_{t+1t} and a_{tt+1}.
Vertex s for $n<s<t+1$: If $\alpha_s=2$ then the only arrows leaving are $c_{ss-1}=1$ and a_{ss+1}. The relation gives $a_{ss+1}=a_{s-1s}$ and by work at previous vertices we know that a_{s-1s} is determined by c_{t+1t} and a_{tt+1}; hence so is a_{ss+1}. Hence assume $\alpha_s>2$ and so the arrows leaving are $k_{v_s},\ldots,k_{u_s},c_{ss-1},a_{ss+1}$. The Step s relations give

$$\begin{aligned} k_{v_s} &= a_{s-1s}\\ k_{v_s+1} &= k_{v_s}A_{0s}=k_{v_s}A_{ts}\\ &\vdots\\ k_{u_s} &= k_{u_s-1}A_{0s}=k_{u_s-1}A_{ts}\\ a_{ss+1} &= k_{u_s}A_{0s}=k_{u_s}A_{ts}\end{aligned}$$

which by work at the previous vertices can all be expressed in terms of c_{t+1t} and a_{tt+1}.
Vertex n: Similar to the above case.
Vertex 0: Only arrows leaving are c_{0n} and a_{01}, both of which are 1.
We now start at vertex t and work clockwise:
Vertex t: If $\alpha_t=2$ then the only arrows leaving are c_{tt-1} and a_{tt+1}; the relations give $c_{tt-1}=a_{tt+1}c_{t+1t}$. Hence assume $\alpha_t>2$ and so the arrows leaving are $k_{v_t},\ldots,k_{u_t},c_{tt-1},a_{tt+1}$. The Step t relations (read backwards) give

$$\begin{aligned} k_{u_t} &= a_{tt+1}c_{t+1t}\\ k_{u_t-1} &= k_{u_t}C_{0t}=k_{u_t}c_{t+1t}\\ &\vdots\\ k_{v_t} &= k_{v_t+1}C_{0t}=k_{v_t+1}c_{t+1t}\\ c_{tt-1} &= k_{v_t}C_{0t}=k_{v_t}c_{tt-1}\end{aligned}$$

which therefore can all be expressed in terms of c_{t+1t} and a_{tt+1}.
Vertex s for $1\leq s<t$: Similar to the above; read the Step s relations backwards and use work at the previous vertices. $\square$

Theorem 4.3. *Keeping α and θ fixed from before,*

$$\operatorname{Rep}(A_{r,a},\alpha)/\!\!/_{\theta}\mathrm{GL}\longrightarrow\mathbb{C}^2/\tfrac{1}{r}(1,a)$$

is a minimal resolution of singularities.

Proof. It is easy to see that W_{t-1} and W_t glue together to give $\mathscr{O}_{\mathbb{P}^1}(-\alpha_t)$ for each $1\leq t\leq n$, so above the singularity there is a string of $\mathbb{P}^1$'s each with self-intersection number ≤ -2. None of these can be contracted to give a smaller resolution. $\square$

Remark 4.4. For finite subgroups of $SL(2,\mathbb{C})$ the above theorem remains true if we replace the fixed θ by an arbitrary generic stability condition [CS98]. However in the $GL(2,\mathbb{C})$ case if we choose a different stability condition it is *not* true in the general that the above theorem holds, since the moduli may have components. A concrete example is $\frac{1}{3}(1,1)$. Thus in the non-Gorenstein case the question of stability is much more subtle.

5. Tilting Bundles

We want to show that the minimal resolution $\tilde{X}$ of the singularity $\mathbb{C}^2/\frac{1}{r}(1,a)$ is derived equivalent to the reconstruction algebra $A_{r,a}$. To do this, we search for a tilting bundle. During the production of this paper this result has been independently proved by Craw [Cra07], who points out that it actually follows immediately from a result of Van den Bergh [Van04b, Thm B]. The proof here uses a simple trick involving an ample line bundle.

Definition 5.1. *Suppose $\mathfrak{T}$ is a triangulated category with small direct sums. An object $C \in \mathfrak{T}$ is called compact if for any set of objects X_i, the natural map*

$$\coprod \operatorname{Hom}(C, X_i) \to \operatorname{Hom}\left(C, \coprod X_i\right)$$

is an isomorphism.

Denote by $\langle \mathfrak{X} \rangle_{\oplus}$ the smallest full triangulated subcategory of $\mathfrak{T}$ closed under infinite sums containing $\mathfrak{X}$. Note this is necessarily closed under direct summands.

Definition 5.2. *Let $\mathfrak{T}$ be a triangulated category with small direct sums. We say $\mathfrak{T}$ is compactly generated if there exists a set of compact objects $\mathfrak{X}$ such that $\langle \mathfrak{X} \rangle_{\oplus} = \mathfrak{T}$.*

Definition 5.3. *A vector bundle $\mathscr{E}$ of finite rank is called a tilting bundle if*
(i) $\operatorname{Ext}^i(\mathscr{E}, \mathscr{E}) = 0$ *for all* $i > 0$
(ii) $\langle \mathscr{E} \rangle_{\oplus} = D(\mathfrak{Qcoh} X)$.

Now to construct such a tilting bundle on the minimal resolution $\tilde{X}$ we have already noted that for any $1 \leq s \leq n$, W_{s-1} and W_s glued together to give the total space of the line bundle $\mathscr{O}_{\mathbb{P}^1}(-\alpha_t) := U_t$. We have local maps to $\mathbb{P}^1$ but not a global map, however we can pull back line bundles on $\mathbb{P}^1$ to line bundles on the U_t and then glue them together; any combinations will patch together because of the particularly nice gluing of the space $\tilde{X}$.

Consider the line bundle $\mathscr{L}_{\delta_j}$ defined to be the pullback of the bundle $\mathscr{O}_{\mathbb{P}^1}(1)$ on U_j and $\mathscr{O}$ on all other U_k. Define

$$\mathscr{E} = \mathscr{O} \oplus \mathscr{L}_{\delta_1} \oplus \ldots \oplus \mathscr{L}_{\delta_n}$$

To obtain a derived equivalence between the minimal resolution $\tilde{X}$ and the reconstruction algebra $A_{r,a}$ we shall prove that $\mathscr{E}$ is a tilting bundle, with endomorphism ring $A_{r,a}$.

It is easy to explicitly write down the sections of each of the summands defining $\mathscr{E}$, and furthermore these coincide with the following full sheaf description:

Definition 5.4. [Esn85] *A sheaf $\mathscr{F}$ on $\tilde{X}$ is called full if*
(i) *$\mathscr{F}$ is locally free*
(ii) *$\mathscr{F}$ is generated by global sections*
(i) $H^1(\tilde{X}, \mathscr{F}^{\vee} \otimes \omega_{\tilde{X}}) = 0$ *where $\omega_{\tilde{X}}$ is the canonical module.*

Denoting the minimal resolution by $\tilde{X} \xrightarrow{\pi} \mathbb{C}^2/G$ then given any Cohen Macaulay module M it is true that

$$\widetilde{M} := \pi^* M/\text{torsion}$$

is a full sheaf. In fact full sheaves are in 1-1 correspondence with indecomposable Cohen-Macaulay modules by work of Esnault [Esn85]; the inverse map is global sections. Denote the functor $\operatorname{Hom}(-, R)$ by * and note that if M is any Cohen-Macaulay module then $M^* = \pi_* \widetilde{M}^{\vee}$. In this new language $\mathscr{L}_{\delta_s} = \widetilde{S_{i_s}}$ and so $\mathscr{E} = \oplus_{p=1}^{n+1} \widetilde{S_{i_p}}$.

The definition of special Cohen-Macaulay module was originally stated in terms of the corresponding full sheaf:

Lemma 5.5. [Wun88] *S_t is a special CM module $\iff$ $H^1(\widetilde{S_t}^{\vee}) = 0$.*

The following lemma shows that the three ways to produce the non-commutative ring all give the same answer.

Lemma 5.6. $\operatorname{End}_{\tilde{X}}(\mathscr{E}) = \operatorname{End}_{\tilde{X}}\left(\oplus_{p=1}^{n+1} \widetilde{S_{i_p}}\right) \cong \operatorname{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{p=1}^{n+1} S_{i_p}\right) \cong A_{r,a}$.

Proof. The last isomorphism is Theorem 3.23. For the first isomorphism note

$$\begin{aligned} \operatorname{End}_{\tilde{X}}(\mathscr{E}) = \operatorname{End}_{\tilde{X}}\left(\oplus_{p=1}^{n+1} \widetilde{S_{i_p}}\right) &\cong \oplus_{p,q=1}^{n+1} \operatorname{Hom}_{\tilde{X}}\left(\widetilde{S_{i_p}}, \widetilde{S_{i_q}}\right) \\ &\cong \oplus_{p,q=1}^{n+1} \operatorname{Hom}_{\mathbb{C}[x,y]^G}(S_{i_p}, S_{i_q}) \\ &\cong \operatorname{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{p=1}^{n+1} S_{i_p}\right) \end{aligned}$$

where the second step follows by e.g. [Cra07, 3.4]. □

Now for every pair p,q with $1\le p,q\le n$ we have a short exact sequence

$$0\longrightarrow \mathscr{O}\longrightarrow \widetilde{S_{i_q}}\oplus\widetilde{S_{i_p}}\longrightarrow \widetilde{S_{i_q}}\otimes\widetilde{S_{i_p}}\longrightarrow 0$$

which after tensoring by $\widetilde{S_{i_p}}^{\vee}$ gives

$$(B_{p,q})\qquad 0\longrightarrow \widetilde{S_{i_p}}^{\vee}\longrightarrow \left(\widetilde{S_{i_p}}^{\vee}\otimes\widetilde{S_{i_q}}\right)\oplus\mathscr{O}\longrightarrow \widetilde{S_{i_q}}\longrightarrow 0\,.$$

Lemma 5.7. $\mathrm{Ext}^r(\mathscr{E},\mathscr{E})=0$ *for all* $r>0$.

Proof. Since the singularity is rational $H^r(\mathscr{O})=0$ for all $r>0$. Further $\widetilde{S_{i_p}}$ is generated by global sections so $H^1(\widetilde{S_{i_p}})=0$, thus using the short exact sequence $B_{p,p}$

$$0\longrightarrow S_{i_p}{}^{\vee}\longrightarrow \mathscr{O}^2\longrightarrow \widetilde{S_{i_p}}\longrightarrow 0$$

and Lemma 5.5 it is true that $H^r(\widetilde{S_{i_p}})=H^r(\widetilde{S_{i_p}}^{\vee})=0$ for all $r>0$.

But using the $B_{p,q}$ together with these facts shows that $H^r(\widetilde{S_{i_p}}^{\vee}\otimes\widetilde{S_{i_q}})=0$ for all $r>0$ and $1\le p,q\le n$. Hence

$$\mathrm{Ext}^r(\mathscr{E},\mathscr{E})=\bigoplus_{p,q=1}^{n+1}H^r(\widetilde{S_{i_p}}^{\vee}\otimes\widetilde{S_{i_q}})=0\text{ for all }r>0.$$

□

Theorem 5.8. *Let $\tilde{X}$ be the minimal resolution of the singularity $\mathbb{C}^2/\frac{1}{r}(1,a)$, let $A_{r,a}$ be the corresponding reconstruction algebra and let $\mathscr{E}$ be the vector bundle as in Section 4. Then*
(i) $\mathbf{R}\mathrm{Hom}(\mathscr{E},-)$ *induces an equivalence between* $D(\mathfrak{Qcoh}\tilde{X})$ *and* $D(\mathfrak{Mod}A_{r,a})$
(ii) *This equivalence restricts to an equivalence between* $D^b(\mathfrak{Qcoh}\tilde{X})$ *and* $D^b(\mathfrak{Mod}A_{r,a})$
(iii) *This equivalence restricts to an equivalence between* $D^b(\mathfrak{coh}\tilde{X})$ *and* $D^b(\mathfrak{mod}A_{r,a})$
(iv) *Since $\tilde{X}$ is smooth, $A_{r,a}$ has finite global dimension.*

Proof. By Lemma 5.6 and Lemma 5.7 we need only prove that $\langle\mathscr{E}\rangle_{\oplus}=D(\mathfrak{Qcoh}\tilde{X})$. By standard GIT arguments X has ample line bundle $\mathscr{L}:=\widetilde{S_{i_1}}\otimes\ldots\otimes\widetilde{S_{i_n}}$ and it is true by [Nee96, 1.10] that $D(\mathfrak{Qcoh}X)=\langle\mathscr{L}^{-\otimes n}:n\in\mathbb{N}\rangle_{\oplus}$ hence it suffices to prove that $\langle\mathscr{E}\rangle_{\oplus}$ contains all negative tensors of the ample line bundle. But using the short exact sequences $B_{p,q}$ together with suitable tensors of them, (which give triangles) this is indeed true: by the sequence $B_{p,p}$ it follows that $\langle\mathscr{E}\rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\vee}$. Now after tensoring $B_{p,p}$ by $\widetilde{S_{i_p}}^{\vee}$ it follows that $\langle\mathscr{E}\rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\otimes -2}$. Continuing in this fashion $\langle\mathscr{E}\rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\otimes -t}$ for all $t\ge 0$ and all i_p. Now by considering the sequence $B_{p,q}$ tensored by $\widetilde{S_{i_q}}^{\vee}$ it follows that $\langle\mathscr{E}\rangle_{\oplus}$ contains $\widetilde{S_{i_p}}^{\vee}\otimes\widetilde{S_{i_q}}^{\vee}$. Continuing in this manner a simple inductive argument shows that $\langle\mathscr{E}\rangle_{\oplus}$ contains $(\widetilde{S_{i_1}}\otimes\ldots\otimes\widetilde{S_{i_n}})^{\otimes -t}$ for all $t\ge 0$. The result is now standard (see e.g. [HdB, 7.6]). □

Denoting by $\langle\mathscr{E}\rangle$ the smallest thick full triangulated subcategory containing $\mathscr{E}$, it is true by Neeman-Ravenel [Nee92] that $\langle\mathscr{E}\rangle$ coincides with the compact objects of $D(\mathfrak{Qcoh}\tilde{X})$. By [Nee96, 2.3] these are precisely the perfect complexes, which since $\tilde{X}$ is smooth are the whole of $D^b(\mathfrak{coh}\tilde{X})$. Thus it is also true that $\langle\mathscr{E}\rangle=D^b(\mathfrak{coh}\tilde{X})$.

6. Homological Considerations

It is well known that the preprojective algebra is a homologically homogeneous ring of global dimension 2. We observed in Theorem 5.8 that for general labels $[\alpha_1,\ldots,\alpha_n]$ the reconstruction algebra of type A also has finite global dimension, thus it is natural to ask its value and whether the homologically homogeneous property holds.

We shall prove in this section that

$$\text{gldim}A_{r,a} = \begin{cases} 2 \text{ if } a = r-1 \\ 3 \text{ else} \end{cases}$$

and so $A_{r,a}$ is homologically homogeneous only when $r = a-1$, ie when $G = \frac{1}{r}(1,a) \leq SL(2,\mathbb{C})$. Furthermore we show that the projective resolutions of the simples in the non-Azumaya locus are determined by the intersection theory.

To do this we must first (for technical reasons, namely the existence of regular primes) prove that the reconstruction algebra is a prime ring. Using primeness we then prove that the Azumaya locus of $A_{r,a}$ coincides with the smooth locus of its centre $Z = \mathbb{C}[x,y]^G$. The idea behind this is that we can then 'ignore' the simples in the Azumaya locus as they correspond to smooth points and so their projective dimensions are easily controlled.

Definition 6.1. *$A = A_{r,a}$ is a noetherian ring module finite over its centre $Z = \mathbb{C}[x,y]^{\frac{1}{r}(1,a)}$. Define*

$$\begin{aligned} \mathscr{A}_A &= \{\mathfrak{m} \in \max Z : A_\mathfrak{m} \text{ is Azumaya over } Z_\mathfrak{m}\} \\ \mathscr{S}_Z &= \{\mathfrak{m} \in \max Z : Z_\mathfrak{m} \text{ is singular}\} \end{aligned}$$

The set $\mathscr{A}_A$ is called the Azumaya locus of A, $\mathscr{S}_Z$ the singular locus of Z.

Lemma 6.2. *For all $\mathfrak{m} \in \mathscr{A}_A$,* $\text{gldim}A_\mathfrak{m} = \text{gldim}Z_\mathfrak{m}$.

Proof. By [MR87, 13.7.5] $\mathfrak{m}A_\mathfrak{m}$ is the Jacobson radical and unique maximal ideal of $A_\mathfrak{m}$. The result is immediate by Silver [Sil67, 4.6,4.9]. □

The hard work in the global dimension proof comes in computing the projective resolutions of the 1-dimensional simples corresponding to the vertices of $A_{r,a}$.

We firstly prove that $A_{r,a}$ is prime: to do this we invoke the standard trick of centrally localising a nonzerodivisor and then showing that the corresponding central localisation is well behaved.

Lemma 6.3. *$\sum_{t=0}^n C_{tt}$ and $\sum_{t=0}^n A_{tt}$ are central elements in $A_{r,a}$ which are nonzerodivisors.*

Proof. Immediate by Theorem 3.23 □

Before the next lemma we need to set some notation. For a ring R and a set Δ of maps between finitely generated projective modules, denote $\Delta^{-1}R$ to be the corresponding universal localisation in the sense of Schofield [Sch85]. If x is a central element in R, denote by $R[x^{-1}]$ the corresponding central localisation.

Lemma 6.4. *Let $x := \sum_{t=0}^n C_{tt}$. Then $A_{r,a}[x^{-1}] \cong M_{n+1}(\mathbb{C}[X^{\pm 1}, Y])$.*

Proof. Set $\sigma = \{c_{0n}, c_{nn-1}, \dots, c_{21}, c_{10}\}$ and $x := \sum_{t=0}^n C_{tt}$. It is true that

$$A_{r,a}[x^{-1}] \cong \sigma^{-1}A_{r,a}$$

by universality: in $\sigma^{-1}A_{r,a}$ the central element x is inverted, and in the ring $A_{r,a}[x^{-1}]$ the elements of σ have been universally inverted. But now (see e.g. [NRS04, 1.1])

$$\sigma^{-1}A_{r,a} \cong M_{n+1}(e_0\sigma^{-1}A_{r,a}e_0)$$

where e_0 is the trivial loop at the zero vertex of $\sigma^{-1}A_{r,a}$, so it suffices to prove that $e_0\sigma^{-1}A_{r,a}e_0 \cong \mathbb{C}[X^{\pm 1}, Y]$. Denote by $\hat{c}_{ii-1}$ the extra arrows in the universal localisation $\sigma^{-1}A_{r,a}$. Now for $0 \leq z \leq n$ denote

$$f_z = \begin{cases} c_{0n}c_{nn-1}\dots c_{z+1z} \text{ if } z > 0 \\ e_0 \text{ if } z = 0 \end{cases} \quad \text{and } g_z = \begin{cases} \hat{c}_{z+1z}\hat{c}_{z+2z+1}\dots \hat{c}_{0n} \text{ if } z > 0 \\ e_0 \text{ if } z = 0 \end{cases}$$

In this notation $e_0\sigma^{-1}A_{r,a}e_0$ is generated as a ring by the set

$$Y = \{f_{tail(b)}b\, g_{head(b)} : b \text{ arrow in } \sigma^{-1}A_{r,a}\}$$

subject to the relations on $\sigma^{-1}A_{r,a}$. Elements of Y are indexed by arrows so for notational ease denote $\mathtt{b}$ to be the element in Y corresponding to the arrow b. We now take the relations on $\sigma^{-1}A_{r,a}$ and multiply by arrows to get relations in terms of the elements of Y; these give all the relations since the arrows we multiply by are invertible. Doing this we see that $e_0\sigma^{-1}A_{r,a}e_0$ is generated as a ring by

$$Y = \{\hat{\mathtt{c}}_{10}, \mathtt{c}_{10} = \mathtt{k}_0, \mathtt{k}_1, \ldots, \mathtt{k}_{u_n-1}, \mathtt{k}_{u_n} = \mathtt{a}_{0n}, \mathtt{a}_{nn-1}, \ldots, \mathtt{a}_{10}\}$$

subject to the following relations (again denote $\mathtt{A}_{0r} = \mathtt{a}_{01}\ldots\mathtt{a}_{r-1r}$):

$$\mathtt{c}_{10}\hat{\mathtt{c}}_{10} = id = \hat{\mathtt{c}}_{10}\mathtt{c}_{10}$$

$$\begin{array}{lll}
\text{Step 1:} & \text{If } \alpha_1 = 2 & \mathtt{c}_{10}\mathtt{a}_{01} = \mathtt{a}_{12} \\
& \text{If } \alpha_1 > 2 & \mathtt{c}_{10}\mathtt{a}_{01} = \mathtt{k}_1, \mathtt{a}_{01}\mathtt{c}_{01} = \mathtt{k}_1 \\
& & \mathtt{k}_s\mathtt{a}_{01} = \mathtt{k}_{s+1}, \mathtt{a}_{01}\mathtt{k}_s = \mathtt{k}_{s+1}\ \forall\ 1 \le s < u_1 \\
& & \mathtt{k}_{u_1}\mathtt{a}_{01} = \mathtt{a}_{12}. \\
& & \vdots \\
\text{Step t:} & \text{If } \alpha_t = 2 & \mathtt{a}_{t-1t} = \mathtt{a}_{tt+1} \\
& \text{If } \alpha_t > 2 & \mathtt{a}_{t-1t} = \mathtt{k}_{v_t}, \mathtt{k}_{v_t} = \mathtt{A}_{0l_{V_t}}\mathtt{k}_{V_t} \\
& & \mathtt{k}_s\mathtt{A}_{0t} = \mathtt{k}_{s+1}, \mathtt{A}_{0t}\mathtt{k}_s = \mathtt{k}_{s+1}\ \forall\ v_t \le s < u_t \\
& & \mathtt{k}_{u_t}\mathtt{A}_{0t} = \mathtt{a}_{tt+1} \\
& & \vdots \\
\text{Step n:} & \text{If } \alpha_n = 2 & \mathtt{a}_{n-1n} = \mathtt{a}_{n0}, \mathtt{a}_{n0} = \mathtt{A}_{0l_{V_n}}\mathtt{k}_{V_n} \\
& \text{If } \alpha_n > 2 & \mathtt{a}_{n-1n} = \mathtt{k}_{v_n}, \mathtt{k}_{v_n} = \mathtt{A}_{0l_{V_n}}\mathtt{k}_{V_n} \\
& & \mathtt{k}_s\mathtt{A}_{0n} = \mathtt{k}_{s+1}, \mathtt{A}_{0n}\mathtt{k}_s = \mathtt{k}_{s+1}\ \forall\ v_n \le s < u_n - 1 \\
& & \mathtt{k}_{u_n-1}\mathtt{A}_{0n} = \mathtt{a}_{n0}, \mathtt{A}_{0n}\mathtt{k}_{u_n-1} = \mathtt{a}_{n0}
\end{array}$$

from which it easy to see that every element of Y can expressed in terms of $\{\mathtt{c}_{10}, \mathtt{a}_{01}, \hat{\mathtt{c}}_{10}\}$, $\mathtt{c}_{10}$ and $\mathtt{a}_{01}$ commute, with $\mathtt{c}_{10}\hat{\mathtt{c}}_{10} = id = \hat{\mathtt{c}}_{10}\mathtt{c}_{10}$. Thus $e_0\sigma^{-1}A_{r,a}e_0 \cong \mathbb{C}[\mathtt{c}_{10}^{\pm1}, \mathtt{a}_{01}]$, the localised commutative polynomial ring in 2 variables. □

The dimension function we deploy on the rings in this paper is the Gelfand-Kirillov dimension GKdim$R$.

Corollary 6.5. *$A_{r,a}$ is a prime ring with* GKdim$A_{r,a} = 2$.

Proof. Suppose $I, J \trianglelefteq A_{r,a}$ with $IJ = 0$. By Lemma 6.3 $x := \sum_{t=0}^{n} C_{tt}$ is central in $A_{r,a}$ and so on localising $I[x^{-1}]J[x^{-1}] = 0$ in $A_{r,a}[x^{-1}]$. But by Lemma 6.4

$$A_{r,a}[x^{-1}] \cong M_n(\mathbb{C}[X^{\pm1}, Y])$$

which is prime and so $I[x^{-1}] = 0$ (say). Hence some power of x annihilates I, which by Lemma 6.3 forces $I = 0$ since x is a nonzerodivisor. Hence $A_{r,a}$ is indeed prime. Now

$$\text{GKdim}A_{r,a} = \text{GKdim}A_{r,a}[x^{-1}] = \text{GKdim}M_n(\mathbb{C}[X^{\pm1}, Y]) = 2$$

by repeated use of [MR87, 8.2.13]. □

Remark: The statement regarding GK dimension can be seen more directly using [MR87, 8.2.2, 8.2.9(i)].

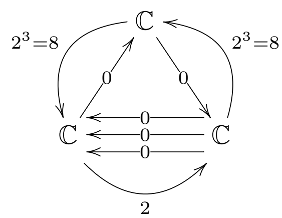
For the remainder of this section for brevity denote $A = A_{r,a}$. Equipped with the above result we can move on to prove that the Azumaya locus of A coincides with the smooth locus of its centre Z (Theorem 6.10). To do this we introduce some canonical simple A-modules: recall

$$\max Z \leftrightarrow \text{ orbits } \mathbb{C}^2/G$$

so for a point $(c, d) \in \mathbb{C}^2$ we denote $\{c, d\}$ to be the maximal ideal of Z corresponding to the orbit containing the point (c, d).

Definition 6.6. *For every point $(c,d) \in \mathbb{C}^2$ define $D^{\{c,d\}}$ to be the representation of $A_{r,a}$ which has a 1-dimensional vector space at every vertex and replaces (in the view of Proposition 3.11) every x by the scalar c, and every y by the scalar d.*

Example 6.7. For the point $(0,2) \in \mathbb{C}^2$ and $A_{7,2}$, $D^{\{0,2\}}$ is

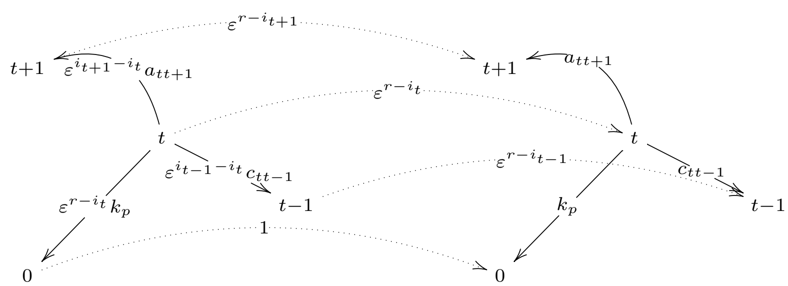


Lemma 6.8. *If (c,d) and (c',d') belong to the same orbit then $D^{\{c,d\}} \cong D^{\{c',d'\}}$. Further if $(c,d) \neq (0,0)$ then $D^{\{c,d\}}$ is simple of dimension $n+1$ with $\mathrm{ann}_Z D^{\{c,d\}} = \{c,d\}$.*

Proof. For the first part we just need to prove that $D^{\{c,d\}} \cong D^{\{\varepsilon c, \varepsilon^a d\}}$. For this define

$$\Phi : D^{\{\varepsilon c, \varepsilon^a d\}} \to D^{\{c,d\}}$$

by $\Phi_t = \varepsilon^{r-i_t}$ where Φ_t maps the t^{th} vertex to the t^{th} vertex. Since maps from vertex t to vertex s are $i_s - i_t$ relative invariants, locally Φ looks like



from which we can see that all relevant diagrams commute and so Φ is map of representations. Since each Φ_i is an isomorphism it follows that Φ is.

To prove $D^{\{c,d\}}$ is simple let N be a proper subrepresentation. Now it is easy to see (where the indices are mod n) that if $c \neq 0$ then

$$(\mathrm{dim} N)_t = 0 \Rightarrow (\mathrm{dim} N)_{t+1} = 0$$

since for N to be a subrepresentation

$$\begin{tikzcd}
t+1 \arrow[r, dotted, "\Psi_{t+1}"] \arrow[d] & t+1 \arrow[d, "c^{i_t - i_{t+1}} \neq 0"] \\
0 \arrow[r, dotted, "0"] & t
\end{tikzcd}$$

needs to commute with Ψ_{t+1} a monomorphism. Similarly if $d \neq 0$ then

$$(\mathrm{dim} N)_t = 0 \Rightarrow (\mathrm{dim} N)_{t-1} = 0.$$

Since N is proper $(\mathrm{dim} N)_t = 0$ for some t and so since $(c,d) \neq (0,0)$ we may use the above to deduce $N = 0$. Hence $D^{\{c,d\}}$ is simple.

To prove that $\mathrm{ann}_Z D^{\{c,d\}} = \{c,d\}$ notice we may, at any vertex, corner the representation $D^{\{c,d\}}$ to give a 1-dimensional representation of Z (i.e. a maximal ideal). It is clear that (for every vertex) this maximal ideal is $\{c,d\}$. Hence the maximal ideal $\{c,d\}$ annihilates each vertex of $D^{\{c,d\}}$ and so annihilates $D^{\{c,d\}}$. $\square$

Remark 6.9. If $(c,d)=(0,0)$ then $D^{\{0,0\}}$ is certainly not simple as it decomposes as $\oplus_{i=0}^{n} D_i$ where D_i is the 1-dimensional simple corresponding to the vertex i. It is clear as in the above proof that $\mathrm{ann}_Z D_i=\{0,0\}$ for all $0\leq i\leq n$ and so $\{0,0\}$ cannot belong to the Azumaya locus of A.

Theorem 6.10. $\mathscr{A}_A=\mathrm{max}Z\backslash\mathscr{S}_Z$.

Proof. Set $\mathrm{PIdeg}A=p$. By Theorem 6.5 A is prime so we may invoke [BG02, III.1.2, III.1.6] to obtain the existence of a maximal ideal M of A such that $M\cap Z:=\mathfrak{m}$ is a maximal ideal of Z and $A_{\mathfrak{m}}$ is Azumaya over $Z_{\mathfrak{m}}$ with
$$A/\mathfrak{m}A\cong M_p(\mathbb{C}).$$
The unique simple $A/\mathfrak{m}A$ module must have dimension p. But by Remark 6.9 $\mathfrak{m}=\{c,d\}$ for some $(c,d)\neq(0,0)$ and so we may use Lemma 6.8 to deduce that $D^{\{c,d\}}$ is a simple $A/\mathfrak{m}A$ module of dimension $n+1$. Hence
$$\mathrm{PIdeg}A=n+1.$$
But now take any $\mathfrak{m}=\{a,b\}\neq\{0,0\}$. The simple A module $D^{\{a,b\}}$ annihilated by $\{a,b\}$ has maximal dimension $n+1=\mathrm{PIdeg}A$ and so by [BG97, 3.1(b)] $\{a,b\}$ belongs to the Azumaya locus. $\square$

We can now proceed to prove the result stated at the beginning of this section regarding global dimension. This involves many preliminary lemmas.

The next lemma is trivial but typical of the arguments used without mention in the more complicated lemmas that follow. It says that if $\alpha_t=2$ for some $1\leq t\leq n$ then viewing everything in the web of paths we have

$$\begin{tikzcd}
t \arrow[r, "a_{tt+1}"] \arrow[d, "c_{tt-1}"'] & t+1 \arrow[d, "c_{t+1t}"] \arrow[ddr, "g"] & \\
t-1 \arrow[r, "a_{t-1t}"] \arrow[drr, "f"'] & t \arrow[dr, dotted, "h"] & \\
& & {}
\end{tikzcd}$$

Lemma 6.11. *Let $\alpha_t=2$ for some $1\leq t\leq n$ and suppose $c_{tt-1}f=a_{tt+1}g$ for some $f\in e_{t-1}A$, $g\in e_{t+1}A$. Then $g=c_{t+1t}h$ and $f=a_{t-1t}h$ for some $h\in e_tA$, where if $t=n$ take $t+1=0$.*

Proof. Via the isomorphism in Theorem 3.23 view everything as polynomials in x and y so that
$$y^{j_{t+1}-j_t}g=a_{tt+1}g=c_{tt-1}f=x^{i_{t-1}-i_t}f.$$
Thus $g=x^{i_{t-1}-i_t}g'=x^{i_t-i_{t+1}}g'=c_{t+1t}g'$ and $f=y^{j_{t+1}-j_t}f'=y^{j_t-j_{t-1}}f'=a_{t-1t}f'$ for some $f',g'\in e_tA$. By uniqueness of path (Lemma 3.22) $f'=g'$. $\square$

Corollary 6.12. *If $\alpha_t=2$ for some $1\leq t\leq n$ then the simple D_t at vertex t has projective resolution*
$$0\longrightarrow e_tA\longrightarrow e_{t-1}A\oplus e_{t+1}A\longrightarrow e_tA\longrightarrow D_t\longrightarrow 0$$
where if $t=n$ take $t+1=0$. Hence $\mathrm{pd}(D_t)=2$.

Proof. Clearly it suffices to prove that the kernel of the map
$$\begin{array}{rcl} e_{t-1}A\oplus e_{t+1}A & \to & e_tA \\ (f,g) & \mapsto & c_{tt-1}f+a_{tt+1}g \end{array}$$
is e_tA. But if (f,g) belongs to the kernel then by Lemma 6.11 $(f,g)=(a_{t-1t},-c_{t+1t})h$ for some $h\in e_tA$. Thus the kernel is $(a_{t-1t},-c_{t+1t})e_tA\cong e_tA$. This means the above is indeed a projective resolution for D_t and so $\mathrm{pd}(D_t)\leq 2$. Since the first syzygy isn't projective $\mathrm{pd}(D_t)=2$. $\square$

Corollary 6.13. *If $\alpha_1 = \ldots = \alpha_n = 2$ then the simple D_0 at vertex 0 has projective resolution*

$$0 \longrightarrow e_0A \longrightarrow e_nA \oplus e_1A \longrightarrow e_0A \longrightarrow D_0 \longrightarrow 0$$

and so $\mathrm{pd}(D_0) = 2$.

Proof. The hypothesis means the quiver is symmetric and so the 0^{th} vertex is indistinguishable from the other vertices. The result now follows from Corollary 6.12 above. $\square$

Now if $\alpha_t > 2$ for some t with $1 \leq t \leq n$ then there are extra maps labelled k leaving vertex t and so the above projective resolutions become more complicated. Unfortunately the labels on these extra k's differ depending on t: the *extra* arrows leaving vertex t with $\alpha_t > 2$ are labelled

$$\begin{array}{ll} \{k_1, \ldots, k_{u_1}\} & \text{for } t = 1 \text{ with } \alpha_1 > 2 \\ \{k_{v_t}, \ldots, k_{u_t}\} & \text{for } 1 < t < n \text{ with } \alpha_t > 2 \\ \{k_{v_n}, \ldots, k_{\sum(\alpha_i - 2)}\} & \text{for } t = n \text{ with } \alpha_n > 2 \end{array}$$

with $k_{v_1} = k_0$ and $k_{u_n} = k_{1+\sum(\alpha_i-2)}$. This had many advantages in previous sections, however here it is more convenient if we don't have this discrepancy and so we introduce yet more notation:

Notation 6.14. *Suppose $\alpha_t > 2$ for some t with $1 \leq t \leq n$. Then denote the **extra** k's leaving the vertex t by $\mathfrak{k}_1, \ldots, \mathfrak{k}_{\alpha_t-2}$ and furthermore denote $\mathfrak{k}_0 = c_{tt-1}$ and $\mathfrak{k}_{\alpha_t-1} = a_{tt+1}$.*

Thus the arrows leaving vertex t (with $\alpha_t > 2$) are now labelled $\mathfrak{k}_0, \ldots, \mathfrak{k}_{\alpha_t-1}$ and the web of paths from vertex t starts (eg for $\alpha_t = 4$):

$$\begin{tikzcd}
t \arrow[rrr, "\mathfrak{k}_3"] \arrow[drr, "\mathfrak{k}_2"] \arrow[ddr, "\mathfrak{k}_1"] \arrow[ddd, "\mathfrak{k}_0"] & & & t+1 \arrow[d, "c_{t+1t}"] \\
& & 0 \arrow[r, "A_{0t}"] \arrow[d, "C_{0t}"] & t \\
& 0 \arrow[r, "A_{0t}"] \arrow[d, "C_{0t}"] & t & \\
t-1 \arrow[r, "a_{t-1t}"] & t & &
\end{tikzcd}$$

Lemma 6.15. *If $\alpha_t > 2$ then the simple D_t at vertex t has projective resolution*

$$0 \longrightarrow (e_tA)^{\alpha_t-1} \longrightarrow (e_{t-1}A) \oplus (e_0A)^{\alpha_t-2} \oplus (e_{t+1}A) \longrightarrow e_tA \longrightarrow D_t \longrightarrow 0$$

and so $\mathrm{pd}(D_t) = 2$.

Proof. It suffices to show that the kernel of the map

$$\begin{array}{rcl} e_{t-1}A \oplus (e_0A)^{\alpha_t-2} \oplus e_{t+1}A & \to & e_tA \\ (g_0, g_1, \ldots, g_{\alpha_t-2}, g_{\alpha_t-1}) & \mapsto & \sum_{i=0}^{\alpha_t-1} \mathfrak{k}_i g_i \end{array}$$

is $(e_tA)^{\alpha_t-1}$. We claim that the kernel is

$$K := (a_{t-1t}, -C_{0t}, 0, \ldots, 0)e_tA + \sum_{i=1}^{\alpha_t-3} (\underbrace{0, \ldots, 0}_{i}, A_{0t}, -C_{0t}, \underbrace{0, \ldots, 0}_{\alpha_t-i-2})e_tA \\ + (0, \ldots, 0, A_{0t}, -c_{t+1t})e_tA$$

where the convention is that the sum is empty if $\alpha_t = 3$. Take (g_i) belonging to the kernel and proceed by induction: if $g_0 = \ldots = g_{\alpha_t-3} = 0$ then the claim is true since $\mathfrak{k}_{\alpha_t-2}g_{\alpha_t-2} = -\mathfrak{k}_{\alpha_t-1}g_{\alpha_t-1}$ and so viewing everything as polynomials in the web of paths, the y component

of g_{α_t-2} must be greater than or equal to j_t and the x component of g_{α_t-1} must be greater than or equal to $i_t - i_{t+1}$. Thus we have

$$\begin{tikzcd}
t \arrow[rr, "\mathfrak{k}_{\alpha_t-1}=a_{tt+1}"] \arrow[dr, "\mathfrak{k}_{\alpha_t-2}"'] & & t+1 \arrow[d, "c_{t+1t}"] \arrow[ddr, "g_{\alpha_t-1}"] & \\
& 0 \arrow[r, "A_{0t}"] \arrow[drr, "g_{\alpha_t-2}"'] & t \arrow[dr, dotted, "r"] & \\
& & & {}
\end{tikzcd}$$

and so by uniqueness of path $g_{\alpha_t-2} = A_{0t}r$ and $g_{\alpha_t-1} = -c_{t+1t}r$ for some $r \in e_tA$. Thus

$$(g_i) = (0, \ldots, 0, g_{\alpha_t-2}, g_{\alpha_t-1}) = (0, \ldots, 0, A_{0t}, -c_{t+1t})r \in K.$$

Thus assume that the claim is true for any $(\underbrace{0, \ldots, 0}_{i}, g_i, \ldots, g_{\alpha_t-1})$ belonging to the kernel with $1 \leq i \leq \alpha_t - 2$; we shall now show that the claim is true for any $(\underbrace{0, \ldots, 0}_{i-1}, g_{i-1}, \ldots, g_{\alpha_t-1})$ belonging to the kernel: certainly

$$\mathfrak{k}_{i-1}g_{i-1} = -\sum_{q=i}^{\alpha_t-1} \mathfrak{k}_q g_q. \tag{1}$$

and so viewing in terms of polynomials the right hand side has y component greater than or equal to the y component of $\mathfrak{k}_i$. Thus

$$\begin{aligned}
y \text{ component of } g_{i-1} \quad &\geq \quad (y \text{ component of } \mathfrak{k}_i) - (y \text{ component of } \mathfrak{k}_{i-1}) \\
&= \quad \begin{cases} j_t & \text{if } i > 1 \\ j_t - j_{t-1} & \text{if } i = 1 \end{cases}
\end{aligned}$$

and so

$$g_{i-1} = \begin{cases} A_{0t}r & \text{if } i > 1 \\ a_{t-1t}r & \text{if } i = 1 \end{cases}$$

for some $r \in e_tA$. Either way

$$\mathfrak{k}_{i-1}g_{i-1} = \begin{cases} \mathfrak{k}_{i-1}A_{0t}r & \text{if } i > 1 \\ c_{tt-1}a_{t-1t}r & \text{if } i = 1 \end{cases} = \mathfrak{k}_i C_{0t} r$$

and so (1) becomes

$$\mathfrak{k}_i(C_{0t}r + g_i) + \sum_{q=i+1}^{\alpha_t-1} \mathfrak{k}_q g_q = 0.$$

By inductive hypothesis

$$(\underbrace{0, \ldots, 0}_{i}, C_{0t}r + g_i, g_{i+1}, \ldots, g_{\alpha_t-1}) \in K$$

but

$$(\underbrace{0, \ldots, 0}_{i-1}, g_{i-1}, \ldots, g_{\alpha_t-1}) =$$

$$\begin{cases} (\underbrace{0, \ldots, 0}_{i-1}, A_{0t}, -C_{0t}, 0, \ldots, 0)r + (\underbrace{0, \ldots, 0}_{i}, C_{0t}r + g_i, g_{i+1}, \ldots, g_{\alpha_t-1}) & \text{if } i > 1 \\ (a_{t-1t}, -C_{0t}, 0, \ldots, 0)r + (0, C_{0t}r + g_1, g_2, \ldots, g_{\alpha_t-1}) & \text{if } i = 1 \end{cases}$$

and so either way $(\underbrace{0, \ldots, 0}_{i-1}, g_{i-1}, \ldots, g_{\alpha_t-1}) \in K$. Thus by induction the kernel is indeed K. But the obvious map $(e_tA)^{\alpha_t-1} \to K$ is an isomorphism. $\square$

Since we have proved the case for vertex t with $1 \leq t \leq n$ we now return to vertex 0:

Lemma 6.16. *If some $\alpha_t > 2$ then the simple D_0 at vertex 0 has projective resolution*

$$0 \longrightarrow \bigoplus_{i=1}^{n}(e_iA)^{\alpha_i-2} \longrightarrow (e_0A)^{1+\sum(\alpha_t-2)} \longrightarrow e_nA \oplus e_1A \longrightarrow e_0A \longrightarrow D_0 \longrightarrow 0$$

and so $\mathrm{pd}(D_0) = 3$.

Proof. Denote $\gamma = \sum_{t=1}^{n}(\alpha_t - 2)$ then by assumption $\gamma \geq 1$. Firstly we prove that the kernel of the map

$$\begin{array}{rcl} e_nA \oplus e_1A & \to & e_0A \\ (f, g) & \mapsto & c_{0n}f + a_{01}g \end{array}$$

(which is $\Omega^2 D_0$) is

$$K_2 := \sum_{t=1}^{1+\gamma} (\hat{C}_{nl_t}k_t, -\hat{A}_{1l_{t-1}}k_{t-1})e_0A.$$

where

$$\hat{C}_{ij} = \left\{ \begin{array}{ll} C_{ij} & i \neq j \\ e_i & i = j \end{array} \right. \quad \text{and} \quad \hat{A}_{ij} = \left\{ \begin{array}{ll} A_{ij} & i \neq j \\ e_i & i = j \end{array} \right.$$

and recall $k_0 = c_{10}$ and $k_{\gamma+1} = a_{n0}$. But this is easy: viewing everything in terms of polynomials in x and y clearly we can assume that f and g are monomials. This being the case if (f, g) belongs to the kernel then $xf = c_{0n}f = -a_{n0}g = -yg$ and so the y-component of f is non-trivial. This means if we view f as a path in the quiver then f must pass through some k_t or a_{tt+1}. By uniqueness of path we may move cycles to the end of the path and thus assume that f passes through the k_t or a_{tt+1} as soon as possible. In particular either $f = k_tf'$ for some $v_n \leq t \leq \sum(\alpha_i - 2) + 1$ or f factors through vertex $n-1$ as $f = c_{nn-1}f''$. Now at vertex $n-1$ if f'' factors through a_{n-1n} then by the relations it is true that $f = k_{v_n}(c_{0n}f'')$ which has been taken care of already. Hence we may assume that f'' factors through $k_{v_{n-1}}, \ldots, k_{u_{n-1}}$ (if they exist) or we move on to the next vertex. Continuing in this way we may ignore $a_{n-1n}, \ldots, a_{01}$ and so f can be written as

$$f = \hat{C}_{nl_t}k_tf'''$$

for some $1 \leq t \leq 1 + \sum(\alpha_i - 2)$, where $f''' \in e_0A$. But now

$$-a_{01}g = c_{0n}f = C_{0l_t}k_tf''' = A_{0l_{t-1}}k_{t-1}f'''$$

and so by uniqueness of path $g = -\hat{A}_{1l_{t-1}}k_{t-1}f'''$. This means

$$(f, g) = (\hat{C}_{nl_t}k_t, -\hat{A}_{1l_{t-1}}k_{t-1})f'''$$

and so the kernel is indeed K_2, as claimed. Thus there is now an obvious surjection

$$(e_0A)^{1+\gamma} \to \Omega^2 D_0 = K_2 = \sum_{t=1}^{1+\gamma} (\hat{C}_{nl_t}k_t, -\hat{A}_{1l_{t-1}}k_{t-1})e_0A$$

and we claim that the kernel of this (which is $\Omega^3 D_0$) is

$$K_3 := \sum_{i=1}^{\gamma} (\underbrace{0, \ldots, 0}_{i-1}, A_{0l_i}, -C_{0l_i}, \underbrace{0, \ldots, 0}_{\gamma-i})e_{l_i}A$$

The proof of this claim is very similar to the proof of Lemma 6.15: proceed by induction. Take $(h_i) = (h_1, \ldots, h_{\gamma+1})$ belonging to the kernel. If $h_1 = \ldots = h_{\gamma-1} = 0$ then

$$\hat{C}_{nl_\gamma}k_\gamma h_\gamma = -\hat{C}_{nl_{\gamma+1}}k_{\gamma+1}h_{\gamma+1} = -a_{n0}h_{\gamma+1}$$

and so viewing everything as polynomials in the web of paths we have

$$\begin{tikzcd}
n \arrow[rr, "k_{\gamma+1}=a_{n0}"] \arrow[d, dashed, "\hat{C}_{nl_\gamma}"'] & & 0 \arrow[dd, "C_{0l_\gamma}"] \arrow[ddr, bend left, "h_{\gamma+1}"] & \\
l_\gamma \arrow[dr, "k_\gamma"] & & & \\
& 0 \arrow[r, "A_{0l_\gamma}"] \arrow[drr, bend right, "h_\gamma"'] & l_\gamma \arrow[dr, dotted, "r"] & \\
& & & {}
\end{tikzcd}$$

where $h_\gamma = A_{0l_\gamma}r$ and $h_{\gamma+1} = -C_{0l_\gamma}r$ for some $r \in e_{l_\gamma}A$ by a similar argument as in Lemma 6.15. Thus

$$(h_i) = (0, \dots, 0, h_\gamma, h_{\gamma+1}) = (0, \dots, 0, A_{0l_\gamma}, -C_{0l_\gamma})r \in K_3$$

and so the claim is true when $h_1 = \dots = h_{\gamma-1} = 0$. Thus assume that the claim is true for any $(\underbrace{0, \dots, 0}_{i}, h_{i+1}, \dots, h_{\gamma+1})$ belonging to the kernel with $1 \leq i \leq \gamma - 1$; we shall now show that the claim is true for any $(\underbrace{0, \dots, 0}_{i-1}, h_i, \dots, h_{\gamma+1})$ belonging to the kernel: certainly

$$\hat{C}_{nl_i}k_ih_i = -\sum_{t=i+1}^{\gamma+1} \hat{C}_{nl_t}k_th_t \tag{2}$$

and so via the same argument as in Lemma 6.15 we see that

$$y \text{ component of } h_i \geq (y \text{ component of } k_{i+1}) - (y \text{ component of } k_i) = j_{l_i}$$

Thus

$$\begin{tikzcd}
0 \arrow[r, "A_{0l_i}"] \arrow[drr, "h_i"'] & l_i \arrow[dr, dotted] & \\
& & {}
\end{tikzcd}$$

i.e. $h_i = A_{0l_i}r$ for some $r \in e_{l_i}A$ and so

$$\hat{C}_{nl_i}k_ih_i = \hat{C}_{nl_i}k_iA_{0l_i}r = \hat{C}_{nl_{i+1}}k_{i+1}C_{0l_i}r.$$

Thus (2) becomes

$$\hat{C}_{nl_{i+1}}k_{i+1}(C_{0l_i}r + h_{i+1}) + \sum_{t=i+2}^{\gamma+1} \hat{C}_{nl_t}k_th_t = 0.$$

But also

$$-\hat{A}_{1l_{i-1}}k_{i-1}h_i = -\hat{A}_{1l_{i-1}}k_{i-1}A_{0l_i}r = -\hat{A}_{1l_i}k_iC_{0l_i}r$$

and so by the inductive hypothesis

$$(\underbrace{0, \dots, 0}_{i}, C_{0l_i}r + h_{i+1}, h_{i+2}, \dots, h_{\gamma+1}) \in K_3.$$

But now

$$\begin{aligned}(\underbrace{0, \dots, 0}_{i-1}, h_i, \dots, h_{\gamma+1}) = (\underbrace{0, \dots, 0}_{i-1}, A_{0l_i}, -C_{0l_i}, 0, \dots, 0)r \\ + (\underbrace{0, \dots, 0}_{i}, C_{0l_i}r + h_{i+1}, h_{i+2}, \dots, h_{\gamma+1})\end{aligned}$$

and so $(\underbrace{0,\dots,0}_{i-1},h_i,\dots,h_{\gamma+1})\in K_3$. Thus by induction the claim is established, so the kernel is K_3. But the obvious map

$$\bigoplus_{i=1}^{\gamma} e_{l_i}A=\bigoplus_{i=1}^{n}(e_iA)^{\alpha_i-2}\to\Omega^3D_0=K_3=\sum_{i=1}^{\gamma}(\underbrace{0,\dots,0}_{i-1},A_{0l_i},-C_{0l_i},\underbrace{0,\dots,0}_{\gamma-i})e_{l_i}A$$

is an isomorphism, hence $pd(D_0)\le 3$. Since the first and second syzygies are not projective $pd(D_0)=3$ □

Summarizing what we have proved

Theorem 6.17. *Consider $G=\frac{1}{r}(1,a)$ and $A=A_{r,a}$. Then for $1\le t\le n$ the simple D_t at vertex t has projective resolution*

$$0\longrightarrow(e_tA)^{\alpha_t-1}\longrightarrow(e_{t-1}A)\oplus(e_0A)^{\alpha_t-2}\oplus(e_{t+1}A)\longrightarrow e_tA\longrightarrow D_t\longrightarrow 0$$

(where if $t=n$ take $t+1=0$) and so $\mathrm{pd}(D_t)=2$. *Further the simple D_0 at vertex 0 has projective resolution*

$$0\longrightarrow\bigoplus_{i=1}^{n}(e_iA)^{\alpha_i-2}\longrightarrow(e_0A)^{1+\sum(\alpha_t-2)}\longrightarrow e_nA\oplus e_1A\longrightarrow e_0A\longrightarrow D_0\longrightarrow 0$$

and so
(i) If $G\le SL(2,\mathbb{C})$ (i.e. all $\alpha_t=2$) then $\mathrm{pd}(D_0)=2$.
(ii) If $G\nleq SL(2,\mathbb{C})$ (i.e. some $\alpha_t>2$) then $\mathrm{pd}(D_0)=3$.

Proof. For $1\le t\le n$ if $\alpha_t=2$ use Corollary 6.12; if $\alpha_t>2$ then use Lemma 6.15. For the 0^{th} vertex use either Corollary 6.13 or Lemma 6.16. □

All the hard work in the global dimension statement has now been done - to finish the proof we use standard ring theoretic methods involving the Azumaya locus

Theorem 6.18.

$$\mathrm{gldim}A_{r,a}=\begin{cases}2 & \textit{if } a=r-1\\ 3 & \textit{else}\end{cases}$$

Proof. By [Rai87]

$$\mathrm{gldim}A=sup\{\mathrm{pd}_AS : S\text{ simple right }R\text{ module}\}.$$

Let S be such a simple and consider ann_ZS; by [BG02, III.1.1(3)] it is a maximal ideal of Z. There are two possibilities
(i) ann_ZS lies in the Azumaya locus. Then

$$\mathrm{pd}_AS=sup\{\mathrm{pd}_{A_\mathfrak{m}}S_\mathfrak{m}:\mathfrak{m}\in\max Z\}=\mathrm{pd}_{A_{\mathrm{ann}_ZS}}S_{\mathrm{ann}_ZS}\le\mathrm{gldim}A_{\mathrm{ann}_ZS}.$$

But by Lemma 6.2 and further since Z_{ann_ZS} is smooth we have

$$\mathrm{gldim}A_{\mathrm{ann}_ZS}=\mathrm{gldim}Z_{\mathrm{ann}_ZS}=\mathrm{Kdim}Z_{\mathrm{ann}_ZS}=\mathrm{ht}(\mathrm{ann}_ZS)=2$$

where the last equality holds since Z is equidimensional [Eis95, 13.4].
(ii) ann_ZS does not lie in the Azumaya locus. In this case by [BG02, III.1.1(3)] and Remark 6.9 S must be one of $D_0,\dots,D_n$. By Theorem 6.17 the projective dimensions of these are either 2 or 3.
Combining (i) and (ii) gives the desired result. □

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